

**Important:**

Answer to all 15 questions.

Write your answers on the answer sheets provided.

For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.

No calculator is allowed.

**Question 1.** Suppose  $x_1, x_2, x_3$  are the roots of polynomial

$$P(x) = x^3 - 6x^2 + 5x + 12.$$

The sum  $|x_1| + |x_2| + |x_3|$  is

(A): 4 (B): 6 (C): 8 (D): 14 (E): None of the above.

*Solution.* The choice is (C).

**Question 2.** How many pairs of positive integers  $(x, y)$  are there, those satisfy the identity

$$2^x - y^2 = 1?$$

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

*Solution.* The choice is (A).

**Question 3.** Suppose  $n^2 + 4n + 25$  is a perfect square. How many such non-negative integers  $n$ 's are there?

(A): 1 (B): 2 (C): 4 (D): 6 (E): None of the above.

*Solution.* The choice is (B).

**Question 4.** Put

$$S = 2^1 + 3^5 + 4^9 + 5^{13} + \dots + 505^{2013} + 506^{2017}.$$

The last digit of  $S$  is

(A): 1 (B): 3 (C): 5 (D): 7 (E): None of the above.

*Solution.* The choice is (E).

**Question 5.** Let  $a, b, c$  be two-digit, three-digit, and four-digit numbers, respectively. Assume that the sum of all digits of number  $a + b$ , and the sum of all digits of  $b + c$  are all equal to 2. The largest value of  $a + b + c$  is

(A): 1099 (B): 2099 (C): 1199 (D): 2199 (E): None of the above.

*Solution.* The choice is (E).

**Question 6.** Find all triples of positive integers  $(m, p, q)$  such that

$$2^m p^2 + 27 = q^3, \quad \text{and } p \text{ is a prime.}$$

*Solution.* By the assumption it follows that  $q$  is odd. We have

$$2^m p^2 = (q - 3)(q^2 + 3q + 9).$$

Remark that  $q^2 + 3q + 9$  is always odd. There are two cases:

*Case 1.*  $q = 2^m p + 3$ . We have

$$q^3 = (2^m p + 3)^3 > 2^m p^2 + 27,$$

which is impossible.

*Case 2.*  $q = 2^m + 3$ . We have

$$q^3 = 2^{3m} + 9 \times 2^{2m} + 27 \times 2^m + 27 = 2^m p^2 + 27,$$

which implies

$$p^2 = 2^{2m} + 9 \times 2^m + 27.$$

If  $m \geq 3$ , then  $2^{2m} + 9 \times 2^m + 27 \equiv 3 \pmod{8}$ , but  $p^2 \equiv 1 \pmod{8}$ . We deduce  $m \leq 3$ . By simple computation we find  $m = 1, p = 7, q = 5$ .

**Question 7.** Determine two last digits of number

$$Q = 2^{2017} + 2017^2.$$

*Solution.* We have

$$\begin{aligned} 2^{2017} &= 2^7 \times (2^{10})^{201} = 128 \times 1024^{201} \\ &\equiv 128 \times (-1)^{201} = -128 \equiv 22 \pmod{25}; \\ 2017^2 &\equiv 14 \pmod{25}. \end{aligned}$$

It follows  $P \equiv 11 \pmod{25}$ , by which two last digits of  $P$  are in the set  $\{11, 36, 61, 86\}$ . In other side,  $P \equiv 1 \pmod{4}$ . This implies  $P \equiv 61 \pmod{100}$ . Thus, the number 61 subjects to the question.

**Question 8.** Determine all real solutions  $x, y, z$  of the following system of equations

$$\begin{cases} x^3 - 3x &= 4 - y \\ 2y^3 - 6y &= 6 - z \\ 3z^3 - 9z &= 8 - x. \end{cases}$$

*Solution.* From  $x^3 + y = 3x + 4$  it follows  $x^3 - 2 - 3x = 2 - y$ . Then

$$(x - 2)(x + 1)^2 = 2 - y \tag{1}$$

By  $2y^3 - 4 - 6y = 2 - z$ , we have

$$2(y - 2)(y + 1)^2 = 2 - z. \tag{2}$$

Similarly, by  $3z^3 - 3 - 3 - 9z = 2 - x$  we have

$$3(z - 2)(z + 1)^2 = (2 - x). \tag{3}$$

Combining (1)-(2)-(3) we obtain

$$(x - 2)(y - 2)(z - 2) \left( (x + 1)^2(y + 1)^2(z + 1)^2 + \frac{1}{6} \right) = 0.$$

Hence,  $(x - 2)(y - 2)(z - 2) = 0$ . Comparing this with (1), (2) and (3), we find the unique solution  $x = y = z = 2$ .

**Question 9.** Prove that the equilateral triangle of area 1 can be covered by five arbitrary equilateral triangles having the total area 2.

*Solution.*

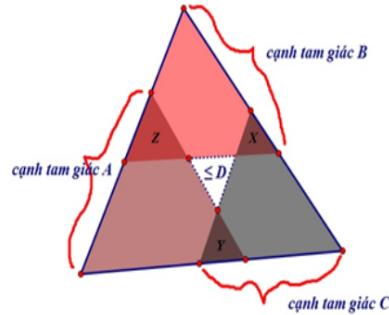


Figure 1: For Question 9

Let  $S$  denote the triangle of area 1. It is clearly that if  $a \geq b$  then triangle of area  $a$  can cover triangle of area  $b$ . It suffices to consider the case when the areas of five small triangles are all smaller than 1. Let  $1 \geq A \geq B \geq C \geq D \geq E$  stand for the areas. We will prove that the sum of side-lengths of  $B$  and  $C$  is not smaller than the side-length of triangle of area 1. Indeed, suppose  $\sqrt{B} + \sqrt{C} < \sqrt{1} = 1$ . It follows

$$2 = A + B + C + D + E < 1 + B + C + 2\sqrt{BC} = 1 + (\sqrt{B} + \sqrt{C})^2 < 2,$$

which is impossible.

We cover  $S$  by  $A, B, C$  as Figure 1. We see that  $A, B, C$  will have common parts, mutually. Suppose

$$X = B \cap C; \quad Y = A \cap C; \quad Z = A \cap B.$$

It follows

$$X + Y \leq C; \quad Y + Z \leq A; \quad Z + X \leq B.$$

We deduce  $A, B, C$  cover a part of area:

$$\begin{aligned} A + B + C - X - Y - Z &\geq A + B + C - \frac{1}{2}[(X + Y) + (Y + Z) + (Z + X)] \\ &\geq \frac{1}{2}(A + B + C) = 1 - \frac{D + E}{2} \geq 1 - D. \end{aligned}$$

Thus,  $D$  can cover the remained part of  $S$ .

**Question 10.** Find all non-negative integers  $a, b, c$  such that the roots of equations:

$$x^2 - 2ax + b = 0; \tag{1}$$

$$x^2 - 2bx + c = 0; \tag{2}$$

$$x^2 - 2cx + a = 0 \tag{3}$$

are non-negative integers.

*Solution.* We see that  $a^2 - b, b^2 - c, c^2 - a$  are perfect squares. Namely,

$$a^2 - b = p^2; \quad b^2 - c = q^2; \quad c^2 - a = r^2.$$

There are two cases:

*Case 1.*  $b = 0$ . We derive that  $b = c = 0$ . Thus  $(a, b, c) = (0, 0, 0)$  is unique solution.

*Case 2.*  $a, b, c \neq 0$ . We have  $a^2 - b \leq (a - 1)^2 = a^2 - 2a + 1$ . This implies  $b \geq 2a - 1$ . Similarly, we can prove that  $c \geq 2b - 1$ , and  $a \geq 2c - 1$ . Combining three above inequalities we deduce  $a + b + c \leq 3$ . By simple computation we obtain  $(a, b, c) = (1, 1, 1)$ .

**Question 11.** Let  $S$  denote a square of the side-length 7, and let eight squares of the side-length 3 be given. Show that  $S$  can be covered by those eight small squares.

*Solution.*

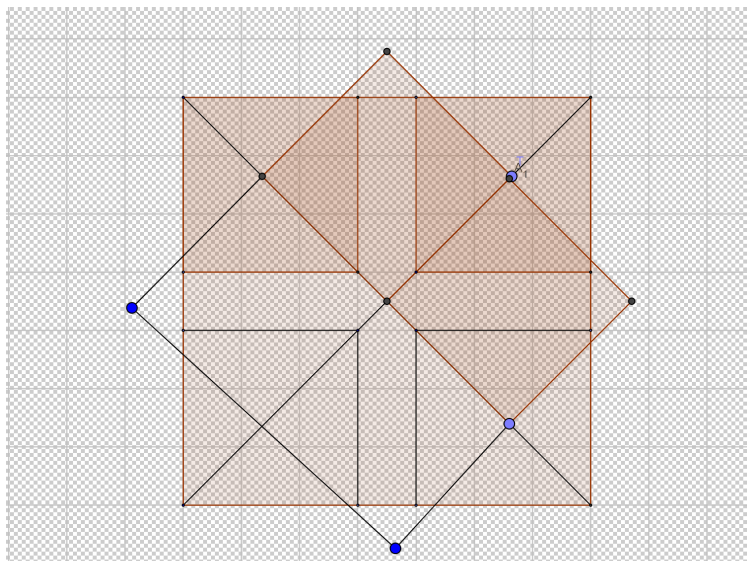


Figure 2: For Question 11

Figure 2 is a solution.

**Question 12.** Does there exist a sequence of 2017 consecutive integers which contains exactly 17 primes?

*Solution.* It is easy to see that there are more than 17 primes in the sequence of numbers  $1, 2, 3, 4, \dots, 2017$ . Precisely, there are 306 primes in that sequence. Remark that if the sequence

$$k + 1, k + 2, \dots, k + 2017$$

was changed by the sequence

$$k, k + 1, \dots, k + 2016,$$

then the numbers of primes in the latter and former sequences are either equal, more or less by 1. In what follows, we say the such change *a shift back with 1 step*. First moment, we consider the sequence of 2017 consecutive integers:

$$2018! + 2, 2018! + 3, \dots, 2018! + 2018$$

which contain no prime. After  $2018!+1$  times shifts back, we obtain the sequence

$$1, 2, 3, 4, \dots, 2017.$$

The last sequence has 306 primes, while the first sequence has no prime. Reminding the above remark we conclude that there is a moment in which the sequence contains exactly 17 primes.



**Question 13.** Let  $a, b, c$  be the side-lengths of triangle  $ABC$  with  $a + b + c = 12$ . Determine the smallest value of

$$M = \frac{a}{b+c-a} + \frac{4b}{c+a-b} + \frac{9c}{a+b-c}.$$

*Solution.* Put

$$x := \frac{b+c-a}{2}, y := \frac{c+a-b}{2}, z := \frac{a+b-c}{2}.$$

Then  $x, y, z > 0$ , and

$$x + y + z = \frac{a+b+c}{2} = 6, \quad a = y + z, \quad b = z + x, \quad c = x + y.$$

We have

$$\begin{aligned} M &= \frac{y+z}{2x} + \frac{4(z+x)}{2y} + \frac{9(x+y)}{2z} = \frac{1}{2} \left[ \left( \frac{y}{x} + \frac{4x}{y} \right) + \left( \frac{z}{x} + \frac{9x}{z} \right) + \left( \frac{4z}{y} + \frac{9y}{z} \right) \right] \\ &\geq \frac{1}{2} \left( 2\sqrt{\frac{y}{x} \cdot \frac{4x}{y}} + 2\sqrt{\frac{z}{x} \cdot \frac{9x}{z}} + 2\sqrt{\frac{4z}{x} \cdot \frac{9y}{z}} \right) = 11. \end{aligned}$$

The equality occurs in the above if and only if

$$\begin{cases} \frac{y}{x} = \frac{4x}{y} \\ \frac{x}{z} = \frac{9x}{z} \\ \frac{4z}{y} = \frac{9y}{z}, \end{cases}$$

or

$$\begin{cases} y = 2x \\ z = 3x \\ 2z = 3y. \end{cases}$$

Since  $x + y + z = 6$  we receive  $x = 1, y = 2, z = 3$ . Thus  $\min S = 11$  if and only if  $(a, b, c) = (5, 4, 3)$ .

**Question 14.** Given trapezoid  $ABCD$  with bases  $AB \parallel CD$  ( $AB < CD$ ). Let  $O$  be the intersection of  $AC$  and  $BD$ . Two straight lines from  $D$  and  $C$  are perpendicular to  $AC$  and  $BD$  intersect at  $E$ , i.e.  $CE \perp BD$  and  $DE \perp AC$ . By analogy,  $AF \perp BD$  and  $BF \perp AC$ . Are three points  $E, O, F$  located on the same line?

*Solution.* Since  $E$  is the orthocenter of triangle  $ODC$ , and  $F$  is the orthocenter of triangle  $OAB$  we see that  $OE$  is perpendicular to  $CD$ , and  $OF$  is perpendicular to  $AB$ . As  $AB$  is parallel to  $CD$ , we conclude that  $E, O, F$  are straightly lined.

**Question 15.** Show that an arbitrary quadrilateral can be divided into nine isosceles triangles.

*Solution.* Figures 3, 4, and 5 shows some solution.

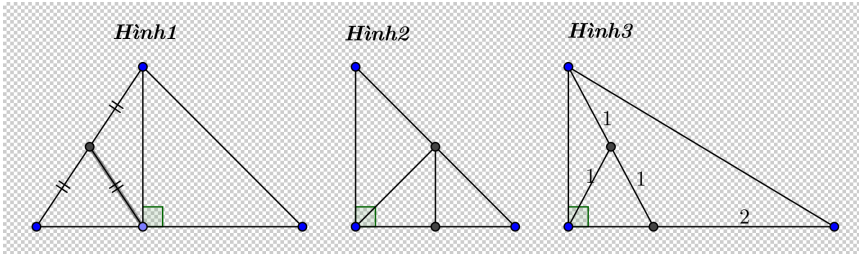


Figure 3: For Question 15

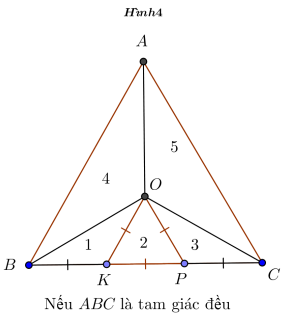


Figure 4: For Question 15

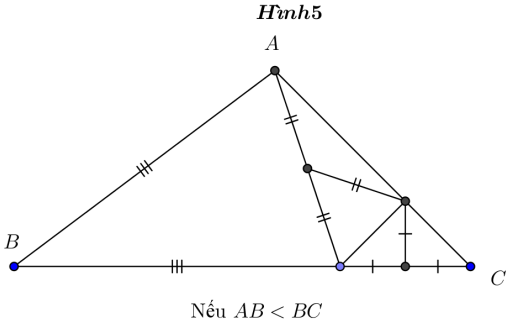


Figure 5: For Question 15

## Important:

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**Question 1.** Suppose  $x_1, x_2, x_3$  are the roots of polynomial  $P(x) = x^3 - 4x^2 - 3x + 2$ .

The sum  $|x_1| + |x_2| + |x_3|$  is

(A): 4 (B): 6 (C): 8 (D): 10 (E): None of the above.

*Solution.* The solution is (B).

**Question 2.** How many pairs of positive integers  $(x, y)$  are there, those satisfy the identity

$$2^x - y^2 = 4?$$

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

*Solution.* The solution is (A).

**Question 3.** The number of real triples  $(x, y, z)$  that satisfy the equation

$$x^4 + 4y^4 + z^4 + 4 = 8xyz$$

is

(A): 0; (B): 1; (C): 2; (D): 8; (E): None of the above.

*Solution.* The solution is (E).

**Question 4.** Let  $a, b, c$  be three distinct positive numbers. Consider the quadratic polynomial

$$P(x) = \frac{c(x-a)(x-b)}{(c-a)(c-b)} + \frac{a(x-b)(x-c)}{(a-b)(a-c)} + \frac{b(x-c)(x-a)}{(b-c)(b-a)} + 1.$$

The value of  $P(2017)$  is

(A): 2015 (B): 2016 (C): 2017 (D): 2018 (E): None of the above.

*Solution.* The solution is (D).

**Question 5.** Write 2017 following numbers on the blackboard:

$$-\frac{1008}{1008}, -\frac{1007}{1008}, \dots, -\frac{1}{1008}, 0, \frac{1}{1008}, \frac{2}{1008}, \dots, \frac{1007}{1008}, \frac{1008}{1008}.$$

One processes some steps as: erase two arbitrary numbers  $x, y$  on the blackboard and then write on it the number  $x + 7xy + y$ . After 2016 steps, there is only one number. The last one on the blackboard is

(A):  $-\frac{1}{1008}$  (B): 0 (C):  $\frac{1}{1008}$  (D):  $-\frac{144}{1008}$  (E): None of the above.

*Solution.* The solution is (D).

**Question 6.** Find all pairs of integers  $a, b$  such that the following system of equations has a unique integral solution  $(x, y, z)$

$$\begin{cases} x + y = a - 1 \\ x(y + 1) - z^2 = b. \end{cases}$$

*Solution.* Write the given system in the form

$$\begin{cases} x + y + 1 = a \\ x(y + 1) - z^2 = b. \end{cases} \quad (*)$$

System (\*) is symmetric by  $x, y + 1$  and is reflect in  $z$  at 0 then the necessary condition for (\*) to have a unique solution is  $(x, y + 1, z) = (t, t, 0)$ . Putting this in (\*), we find  $a^2 = 4b$ . Conversely, if  $a^2 = 4b$  then

$$(x - (y + 1))^2 + 4z^2 = (x + y + 1)^2 + 4z^2 - 4x(y + 1) = a^2 - 4b = 0.$$

This implies the system has a unique solution

$$(x, y + 1, z) = \left(\frac{a}{2}, \frac{a}{2}, 0\right).$$

**Question 7.** Let two positive integers  $x, y$  satisfy the condition  $x^2 + y^2 \vdots 44$ . Determine the smallest value of  $T = x^3 + y^3$ .

*Solution.* Note that  $44 = 4 \times 11$ . It follows  $x^2 + y^2 \vdots 11$ . We shall prove that  $x \vdots 11$  and  $y \vdots 11$ . Indeed, if  $x$  and  $y$  are not divisible by 11 then by the Fermat's little theorem, we have

$$x^{10} + y^{10} \equiv 2 \pmod{11}. \quad (1)$$

On the other hand, since  $x^2 + y^2 \vdots 11$  then  $x^2 + y^2 \equiv 0 \pmod{11}$ . It follows

$$x^{10} + y^{10} \equiv 0 \pmod{11},$$

which is not possible by (1). Hence,  $x \vdots 11$  or  $y \vdots 11$  and that follow  $x \vdots 11$  and  $y \vdots 11$  simultaneously (if  $x \vdots 11$  then from  $x^2 + y^2 \vdots 11$ . It follows  $y^2 \vdots 11$  and then  $y \vdots 11$ ). In other side, we have  $x^2 + y^2 \vdots 4$  and  $x^2 \equiv 0, 1 \pmod{4}, y^2 \equiv 0, 1 \pmod{4}$ . We then have  $x^2 \equiv 0 \pmod{4}, y^2 \equiv 0 \pmod{4}$ . It follows  $x \vdots 2, y \vdots 2$ . Since  $(2, 11) = 1$ ,  $x \vdots (22)$  and  $y \vdots (22)$ . Thus,  $\min A = (22)^3 + (22)^3 = 21296$ .

**Question 8.** Let  $a, b, c$  be the side-lengths of triangle  $ABC$  with  $a + b + c = 12$ . Determine the smallest value of

$$M = \frac{a}{b+c-a} + \frac{4b}{c+a-b} + \frac{9c}{a+b-c}.$$

*Solution.* Let  $x = \frac{b+c-a}{2}, y = \frac{c+a-b}{2}, z = \frac{a+b-c}{2}$  then  $x, y, z > 0$  and

$x + y + z = \frac{a+b+c}{2} = 6, a = y+z, b = z+x, c = x+y$ . We have

$$\begin{aligned} M &= \frac{y+z}{2x} + \frac{4(z+x)}{2y} + \frac{9(x+y)}{2z} = \frac{1}{2} \left[ \left( \frac{y}{x} + \frac{4x}{y} \right) + \left( \frac{z}{x} + \frac{9x}{z} \right) + \left( \frac{4z}{y} + \frac{9y}{z} \right) \right] \\ &\geq \frac{1}{2} \left( 2\sqrt{\frac{y}{x} \cdot \frac{4x}{y}} + 2\sqrt{\frac{z}{x} \cdot \frac{9x}{z}} + 2\sqrt{\frac{4z}{y} \cdot \frac{9y}{z}} \right) = 11. \end{aligned}$$

The equality yields if and only if

$$\left\{ \begin{array}{l} \frac{y}{x} = \frac{4x}{y} \\ \frac{z}{x} = \frac{9x}{z} \\ \frac{4z}{y} = \frac{9y}{z} \end{array} \right.$$

Equivalently,

$$\left\{ \begin{array}{l} y = 2x \\ z = 3x \\ 2z = 3y. \end{array} \right.$$

By simple computation we receive  $x = 1, y = 2, z = 3$ . Therefore,  $\min S = 11$  when  $(a, b, c) = (5, 4, 3)$ .

**Question 9.** Cut off a square carton by a straight line into two pieces, then cut one of two pieces into two small pieces by a straight line, ect. By cutting 2017 times we obtain 2018 pieces. We write number 2 in every triangle, number 1 in every quadrilateral, and 0 in the polygons. Is the sum of all inserted numbers always greater than 2017?

*Solution.* After 2017 cuts, we obtain 2018  $n$ -convex polygons with  $n \geq 3$ . After each cut the total of all sides of those  $n$ -convex polygons increases at most 4. We deduce that the total number of sides of 2018 pieces is not greater than  $4 \times 2018$ . If the side of a piece is  $k_j$ , then the number inserted on it is greater or equal to  $5 - k_j$ . Therefore, the total of all inserted numbers on the pieces is greater or equal to

$$\sum_j (5 - k_j) = 5 \times 2018 - \sum_j k_j \geq 5 \times 2018 - 4 \times 2018 = 2018 > 2017.$$

The answer is positive.



**Question 10.** Consider all words constituted by eight letters from  $\{C, H, M, O\}$ . We arrange the words in an alphabet sequence. Precisely, the first word is CCCC-CCCC, the second one is CCCCCCCH, the third is CCCCCCCM, the fourth one is CCCCCCO, ..., and the last word is OOOOOOOO.

- a) Determine the 2017<sup>th</sup> word of the sequence?
- b) What is the position of the word HOMCHOMC in the sequence?

*Solution.* We can associate the letters  $C, H, M, O$  with four numbers 0, 1, 2, 3, respectively. Thus, the arrangement of those words as a dictionary is equivalent to arrangement of those numbers increasing.

- a) Number 2017 in quaternary is  $\{133201\}_4 = \{00133201\}_4 \sim CCHOOMCH$ .
- b) The word  $HOMCHOMC$  is corresponding to the number  $\{13201320\}_4$  which means the number 13201320 in quaternary. Namely,

$$\{13201320\}_4 = 4^7 + 3 \times 4^6 + 2 \times 4^5 + 0 \times 4^4 + 1 \times 4^3 + 3 \times 4^2 + 2 \times 4 + 0.$$

A simple computation gives  $\{13201320\}_4 = 30840$ . Thus, the word  $HOMCHOMC$  is 30840<sup>th</sup> in the sequence.

**Question 11.** Let  $ABC$  be an equilateral triangle, and let  $P$  stand for an arbitrary point inside the triangle. Is it true that

$$\left| \widehat{PAB} - \widehat{PAC} \right| \geq \left| \widehat{PBC} - \widehat{PCB} \right|?$$

*Solution.* If  $P$  lies on the symmetric straightline  $Ax$  of  $\Delta ABC$ , then

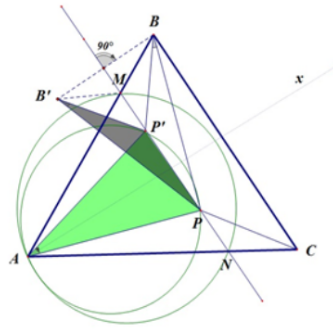


Figure 6: For Question 11

$$\left| \widehat{PAB} - \widehat{PAC} \right| = \left| \widehat{PBC} - \widehat{PCB} \right|.$$

We should consider other cases. Let  $P'$  denote the symmetric point of  $P$  with respect to  $Ax$ . The straightline  $PP'$  intersects  $AB$  and  $AC$  at  $M$  and  $N$ , respectively. Choose  $B'$  that is symmetric point of  $B$  with respect to  $MN$ . Then

$$\left| \widehat{PAB} - \widehat{PAC} \right| = \widehat{PAP'},$$

and

$$\left| \widehat{PBC} - \widehat{PCB} \right| = \widehat{PBP'} = \widehat{PB'P'}.$$

We will prove that

$$\widehat{PAP'} \geq \widehat{PB'P'}. \quad (*)$$

Indeed, consider the circumscribed circle ( $O$ ) of the equilateral triangle  $AMN$ . Since

$$\widehat{MB'N} = \widehat{MBN} \leq \widehat{MBC} = \widehat{MAN} = 60^\circ,$$

$B'$  is outside ( $O$ ). Consider the circumscribed circle ( $O'$ ) of the equilateral triangle  $APP'$ . It is easy to see that ( $O'$ ) inside ( $O$ ), by which  $B'$  is outside ( $O'$ ). Hence,  $\widehat{PAP'} \geq \widehat{PB'P'}$ . The inequality (\*) is proved.

**Question 12.** Let  $(O)$  denote a circle with a chord  $AB$ , and let  $W$  be the midpoint of the minor arc  $AB$ . Let  $C$  stand for an arbitrary point on the major arc  $AB$ . The tangent to the circle  $(O)$  at  $C$  meets the tangents at  $A$  and  $B$  at points  $X$  and  $Y$ , respectively. The lines  $WX$  and  $WY$  meet  $AB$  at points  $N$  and  $M$ , respectively.

Does the length of segment  $NM$  depend on position of  $C$ ?

*Solution.* Let  $T$  be the common point of  $AB$  and  $CW$ . Consider circle  $(Q)$  touching  $XY$  at  $C$  and touching  $AB$  at  $T$ .

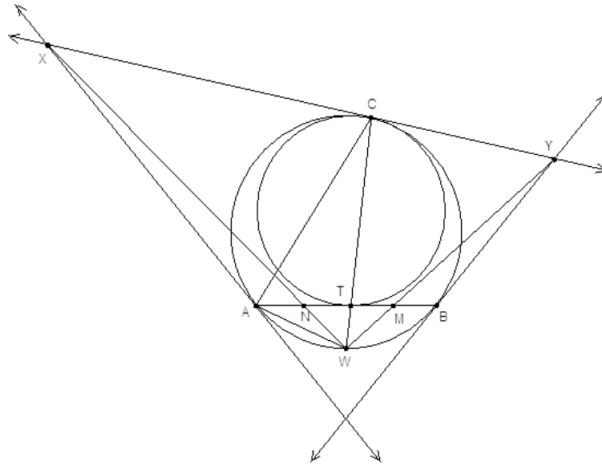


Figure 7: For Question 12

Since

$$\widehat{ACW} = \widehat{WAT} \quad \left( = \frac{1}{2}\widehat{AW} = \frac{1}{2}\widehat{WB} \right)$$

and  $\widehat{AWT} = \widehat{CWA}$ , we obtain that  $\triangle AWT, \triangle CWA$  are similar triangles. Then

$$WA^2 = WT \times WC.$$

It is easy to see that  $WX$  is the radical axis of  $A$  and  $(Q)$ , thus it passes through the midpoint  $N$  of segment  $AT$ . Similarly,  $WY$  passes through the midpoint  $M$  of segment  $BT$ . We deduce  $MN = \frac{AB}{2}$ .

**Question 13.** Let  $ABC$  be a triangle. For some  $d > 0$  let  $P$  stand for a point inside the triangle such that

$$|AB| - |PB| \geq d, \text{ and } |AC| - |PC| \geq d.$$

Is the following inequality true

$$|AM| - |PM| \geq d,$$

for any position of  $M \in BC$ ?

*Solution.* Note that  $AM$  always intersects  $PB$  or  $PC$  of  $\triangle PBC$ . Without loss of generality, assume that  $AM$  has common point with  $PB$ . Then  $ABMP$  is a convex quadrilateral with diagonals  $AM$  and  $PB$ .

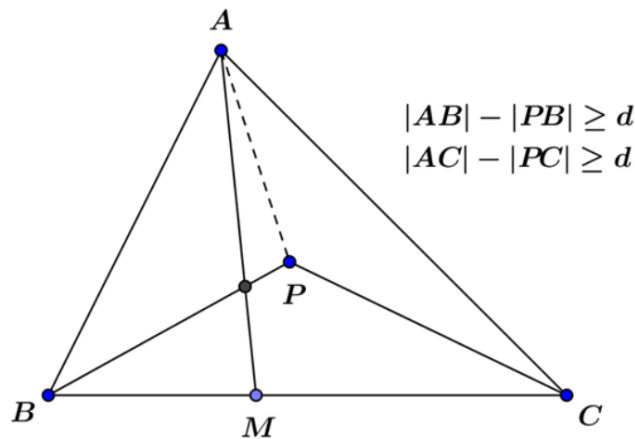


Figure 8: For Question 13

It is known that for every convex quadrilateral, we have

$$|AM| + |PB| \geq |AB| + |PM|,$$

that follows

$$|AM| - |PM| \geq |AB| - |PB| \geq d.$$

**Question 14.** Put

$$P = m^{2003}n^{2017} - m^{2017}n^{2003}, \quad \text{where } m, n \in \mathbb{N}.$$

- a) Is  $P$  divisible by 24?
- b) Do there exist  $m, n \in \mathbb{N}$  such that  $P$  is not divisible by 7?

*Solution.* We have

$$P = m^{2003}n^{2013}(n^{14} - m^{14}) = m^{2003}n^{2013}(n^7 - m^7)(n^7 + m^7).$$

It is easy to prove  $P$  is divisible by 8, and by 3.

b) It suffices to choose  $m, n$  such that the remainders of those divided by 7 are not 0 and distinct. For instance,  $m = 2$  and  $n = 1$ .

**Question 15.** Let  $S$  denote a square of side-length 7, and let eight squares with side-length 3 be given. Show that it is impossible to cover  $S$  by those eight small squares with the condition: an arbitrary side of those (eight) squares is either coincided, parallel, or perpendicular to others of  $S$ .

*Solution.* Let  $ABCD$  be the square  $S$ , and  $M, N, P, Q$  be the midpoints of sides of  $S$ , and  $O$  is the center of  $S$ . Consider nine points:  $A, B, C, D, M, N, P, Q, O$ . Each square of the side-length 3 satisfied the condition cover at most one of those nine points. The proof is complete.