Hanoi Open Mathematical Competition 2017

Junior Section

Saturday, 4 March 2017 08h30-11h30

Important:

Answer to all 15 questions.
Write your answers on the answer sheets provided.
For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.
No calculator is allowed.

Question 1. Suppose x_1, x_2, x_3 are the roots of polynomial

$$P(x) = x^3 - 6x^2 + 5x + 12.$$

The sum $|x_1| + |x_2| + |x_3|$ is
(A): 4 (B): 6 (C): 8 (D): 14 (E): None of the above.

Question 2. How many pairs of positive integers (x, y) are there, those satisfy the identity

$$2^x - y^2 = 1?$$

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

Question 3. Suppose $n^2 + 4n + 25$ is a perfect square. How many such non-negative integers n's are there?
(A): 1 (B): 2 (C): 4 (D): 6 (E): None of the above.

Question 4. Put

$$S = 2^1 + 3^5 + 4^9 + 5^{13} + \cdots + 505^{2013} + 506^{2017}.$$

The last digit of S is
(A): 1 (B): 3 (C): 5 (D): 7 (E): None of the above.

Question 5. Let a, b, c be two-digit, three-digit, and four-digit numbers, respectively. Assume that the sum of all digits of number $a + b$, and the sum of all digits of $b + c$ are all equal to 2. The largest value of $a + b + c$ is
(A): 1099 (B): 2099 (C): 1199 (D): 2199 (E): None of the above.

Question 6. Find all triples of positive integers (m, p, q) such that

$$2^m p^2 + 27 = q^3, \quad \text{and } p \text{ is a prime.}$$

Question 7. Determine two last digits of number

$$Q = 2^{2017} + 2017^2.$$

Question 8. Determine all real solutions x, y, z of the following system of equations

$$\begin{cases} x^3 - 3x &= 4 - y \\ 2y^3 - 6y &= 6 - z \\ 3z^3 - 9z &= 8 - x. \end{cases}$$

Question 9. Prove that the equilateral triangle of area 1 can be covered by five arbitrary equilateral triangles having the total area 2.

Question 10. Find all non-negative integers a, b, c such that the roots of equations:

$$x^2 - 2ax + b = 0; \quad (1)$$
$$x^2 - 2bx + c = 0; \quad (2)$$
$$x^2 - 2cx + a = 0 \quad (3)$$

are non-negative integers.

Question 11. Let S denote a square of the side-length 7, and let eight squares of the side-length 3 be given. Show that S can be covered by those eight small squares.

Question 12. Does there exist a sequence of 2017 consecutive integers which contains exactly 17 primes?

Question 13. Let a, b, c be the side-lengths of triangle ABC with $a+b+c = 12$. Determine the smallest value of

$$M = \frac{a}{b+c-a} + \frac{4b}{c+a-b} + \frac{9c}{a+b-c}.$$

Question 14. Given trapezoid $ABCD$ with bases $AB \parallel CD$ ($AB < CD$). Let O be the intersection of AC and BD. Two straight lines from D and C are perpendicular to AC and BD intersect at E, i.e. $CE \perp BD$ and $DE \perp AC$. By analogy, $AF \perp BD$ and $BF \perp AC$. Are three points E, O, F located on the same line?

Question 15. Show that an arbitrary quadrilateral can be divided into nine isosceles triangles.

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Senior Section

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No calculator is allowed.

Question 1. Suppose x_1, x_2, x_3 are the roots of polynomial $P(x) = x^3 - 4x^2 - 3x + 2$. The sum $|x_1| + |x_2| + |x_3|$ is
(A): 4 (B): 6 (C): 8 (D): 10 (E): None of the above.

Question 2. How many pairs of positive integers (x, y) are there, those satisfy the identity

$$2^x - y^2 = 4?$$

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

Question 3. The number of real triples (x, y, z) that satisfy the equation

$$x^4 + 4y^4 + z^4 + 4 = 8xyz$$

is
(A): 0; (B): 1; (C): 2; (D): 8; (E): None of the above.

Question 4. Let a, b, c be three distinct positive numbers. Consider the quadratic polynomial

$$P(x) = \frac{c(x-a)(x-b)}{(c-a)(c-b)} + \frac{a(x-b)(x-c)}{(a-b)(a-c)} + \frac{b(x-c)(x-a)}{(b-c)(b-a)} + 1.$$

The value of $P(2017)$ is
(A): 2015 (B): 2016 (C): 2017 (D): 2018 (E): None of the above.

Question 5. Write 2017 following numbers on the blackboard:

$$-\frac{1008}{1008}, -\frac{1007}{1008}, \ldots, -\frac{1}{1008}, 0, \frac{1}{1008}, \frac{2}{1008}, \ldots, \frac{1007}{1008}, \frac{1008}{1008}.$$

One processes some steps as: erase two arbitrary numbers x, y on the blackboard and then write on it the number $x + 7xy + y$. After 2016 steps, there is only one number. The last one on the blackboard is

(A): $-\dfrac{1}{1008}$ (B): 0 (C): $\dfrac{1}{1008}$ (D): $-\dfrac{144}{1008}$ (E): None of the above.

Question 6. Find all pairs of integers a, b such that the following system of equations has a unique integral solution (x, y, z)

$$\begin{cases} x + y = a - 1 \\ x(y+1) - z^2 = b. \end{cases}$$

Question 7. Let two positive integers x, y satisfy the condition $x^2 + y^2 \vdots 44$. Determine the smallest value of $T = x^3 + y^3$.

Question 8. Let a, b, c be the side-lengths of triangle ABC with $a+b+c = 12$. Determine the smallest value of

$$M = \frac{a}{b+c-a} + \frac{4b}{c+a-b} + \frac{9c}{a+b-c}.$$

Question 9. Cut off a square carton by a straight line into two pieces, then cut one of two pieces into two small pieces by a straight line, ect. By cutting 2017 times we obtain 2018 pieces. We write number 2 in every triangle, number 1 in every quadrilateral, and 0 in the polygons. Is the sum of all inserted numbers always greater than 2017?

Question 10. Consider all words constituted by eight letters from $\{C, H, M, O\}$. We arrange the words in an alphabet sequence. Precisely, the first word is CCCCCCCC, the second one is CCCCCCCH, the third is CCCCCCCM, the fourth one is CCCCCCCO,..., and the last word is OOOOOOOO.

a) Determine the 2017^{th} word of the sequence?

b) What is the position of the word HOMCHOMC in the sequence?

Question 11. Let ABC be an equilateral triangle, and let P stand for an arbitrary point inside the triangle. Is it true that

$$\left|\widehat{PAB} - \widehat{PAC}\right| \geq \left|\widehat{PBC} - \widehat{PCB}\right|?$$

Question 12. Let (O) denote a circle with a chord AB, and let W be the midpoint of the minor arc AB. Let C stand for an arbitrary point on the major arc AB. The tangent to the circle (O) at C meets the tangents at A and B at points X and Y, respectively. The lines WX and WY meet AB at points N and M, respectively.

Does the length of segment NM depend on position of C?

Question 13. Let ABC be a triangle. For some $d > 0$ let P stand for a point inside the triangle such that

$$|AB| - |PB| \geq d, \text{ and } |AC| - |PC| \geq d.$$

Is the following inequality true

$$|AM| - |PM| \geq d,$$

for any position of $M \in BC$?

Question 14. Put

$$P = m^{2003}n^{2017} - m^{2017}n^{2003}, \quad \text{where} \quad m, n \in \mathbb{N}.$$

a) Is P divisible by 24?
b) Do there exist $m, n \in \mathbb{N}$ such that P is not divisible by 7?

Question 15. Let S denote a square of side-length 7, and let eight squares with side-length 3 be given. Show that it is impossible to cover S by those eight small squares with the condition: an arbitrary side of those (eight) squares is either coincided, parallel, or perpendicular to others of S.