Integer Sequences

Lecturerer - Andrei Jorza

If (x_n) is a sequence so that we have the recurrence $x_{n+2} = ux_{n+1} + vx_n$ then we construct the characteristic equation of the recurrence:

$$x^2 - ux - v = 0$$

with roots a, b. Note that $x_n = a^n$ and $x_n = b^n$ satisfy the given recurrence because $a^2 = ua + v, b^2 = ub + v$. Also if $a = b$ then $a = b = \frac{u}{2}$ and so $x_n = na^n$ also satisfies the recurrence.

If $a \neq b$ then find $c_1, c_2 \in \mathbb{R}$ so that $x_1 = c_1 a + c_2 b, x_2 = c_1 a^2 + c_2 b^2$. Then we know that $x_n = c_1 a^n + c_2 b^n$ verifies the recurrence. Also x_1, x_2 uniquely determine both the constants c_1, c_2 and the sequence x_n. Therefore the sequence satisfying the given recurrence has to be $x_n = c_1 a^n + c_2 b^n$ with c_1, c_2 determined above.

Similarly if $a = b$ then the sequence has to be $x_n = c_1 a^n + c_2 n b^n$ where c_1, c_2 are determined from $x_1 = c_1 a + c_2 b$ and $x_2 = c_1 a^2 + 2c_2 b^2$.

We are going to use the following property of the quadratic extanesions of the rationals:

Lemma 1 *If d is square free and $x \in \mathbb{Q}[\sqrt{d}]$ so that $x^2 \in \mathbb{Z}$ then $x \in \mathbb{Z}$.*

Proof: Let $x = p + q\sqrt{d}$ with $p, q \in \mathbb{Q}$. Then $x^2 = p^2 + dq^2 + 2pq\sqrt{d} \in \mathbb{Z}$. So $pq = 0$. If $p = 0$ then $dq^2 \in \mathbb{Z}$ and this means that the denominator of q^2 has to divide d. But d is square-free so this cannot happen unless the denominator of q is 1, i.e. $q \in \mathbb{Z}$. Similarly if we assume that $q = 0$ then $p \in \mathbb{Z}$. So $x \in \mathbb{Z}[\sqrt{d}]$.

Problems

1. Consider the sequence (a_n) defined by $a_0 = a_1 = 1$ and $a_{n+1} = 14a_n - a_{n-1}$ for $n \geq 1$. Prove that $2a_n - 1$ is a perfect square. (Titu Andreescu, Romania 2002)

2. An integer sequence is defined by $a_0 = 0, a_1 = 1, a_{n+2} = 2a_{n+1} + a_n$ for $n \geq 0$. Prove that $2^k | a_n \iff 2^k | n$. (Shortlist 1988)

Proposed Problems

1. Show that for any positive integer n the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k} 2^{2n-2k} 3^k$$

is the sum of two consecutive squares. (Romania 1999)

2. Consider the sequence $a_1 = 20, a_2 = 30, a_{n+1} = 3a_n - a_{n-1}$ for $n \geq 2$. Find all n so that $1 + 5a_n a_{n+1}$ is a perfect square. (BMO 2002)

3. Let $a_0 = a_1 = 5$ and
$$a_n = \frac{a_{n-1} + a_{n+1}}{98}$$
Prove that $\frac{a_n+1}{6}$ is a perfect square. (Titu, Razvan, Mathematical Olympiad Challenges)

4. Consider the sequence $a_1 = 1$ and for $n \geq 1$ we have
$$a_{n+1} = 2a_n + \sqrt{3a_n^2 - 3}$$
Prove that $a_{3n+1} = a_{n+1}(a_{n+1}^2 + 1)$.

5. Consider the sequence $a_0 = 1, a_1 = 2$ and $a_{n+1} = 4a_n + a_{n-1}$. Prove that $\frac{a_{6n}}{a_{2n}}$ is not the cube of a rational number.

6. Let $a_0 = 4, a_1 = 22$ and $a_{n+1} = 6a_n - a_{n-1}$. Prove that one can write a_n as
$$a_n = \frac{y_n^2 + 7}{y_n - x_n}$$
with $x_n, y_n \in \mathbb{N}$. (MOSP 2001)

7. Let $c \in \mathbb{N}^*$. Let $x_1 = 1, x_2 = c$ and $x_{n+1} = 2x_n - x_{n-1} + 2$ for $n \geq 2$. Prove that for any k there is a r so that $x_k x_{k+1} = x_r$. (Shortlist 1984)

8. Let $a, b \in \mathbb{Z}$. Define (a_n) by $a_0 = a, a_1 = b, a_2 = 2b-a+2$ and $a_{n+3} = 3a_{n+2}-3a_{n+1}+a_n$. Find a, b so that a_n is a perfect square for $n \geq 1998$. (Vietnam 1998)

9. Consider the sequence $x_0 = 2, x_1 = 503, x_{n+2} = 503x_{n+1} - 1996x_n$. For a positive integer k choose integers $s_1, \ldots, s_k \geq k$ and let p_i be a prime divisor of $f(2^{s_i})$. Prove that $\sum p_i | 2^t \iff k | 2^t$. (Vietnam 1997)

Not really related as proof but still looking similar as statement:

1. Consider the sequence (a_n) so that $a_1 = 43, a_2 = 142, a_{n+1} = 3a_n + a_{n-1}$. Prove that $\gcd(a_n, a_{n+1}) = 1$. Prove that for any $m \in \mathbb{N}$ there are infinitely many n so that $m | \gcd(a_n - 1, a_{n+1} - 1)$. (Bulgaria 2000)

2. Let $x_1, x_2 \in \mathbb{N}$. Define x_{n+2} as the smallest/greatest prime divisor of $x_n + x_{n+1}$. Prove that x_n is eventually periodic. (Romania 2002)

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jorza@fas.harvard.edu
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