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Chapter 1

Symmetric Rational Inequalities

1.1 Applications

1.1. If $a$, $b$, $c$ are nonnegative real numbers, then

$$\frac{a^2 - bc}{3a + b + c} + \frac{b^2 - ca}{3b + c + a} + \frac{c^2 - ab}{3c + a + b} \geq 0.$$

1.2. If $a$, $b$, $c$ are positive real numbers, then

$$\frac{4a^2 - b^2 - c^2}{a(b + c)} + \frac{4b^2 - c^2 - a^2}{b(c + a)} + \frac{4c^2 - a^2 - b^2}{c(a + b)} \leq 3.$$

1.3. Let $a$, $b$, $c$ be nonnegative real numbers, no two of which are zero. Prove that

(a) \[ \frac{1}{a^2 + bc} + \frac{1}{b^2 + ca} + \frac{1}{c^2 + ab} \geq \frac{3}{ab + bc + ca}; \]

(b) \[ \frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq \frac{2}{ab + bc + ca}. \]

(c) \[ \frac{1}{a^2 + 2bc} + \frac{1}{b^2 + 2ca} + \frac{1}{c^2 + 2ab} > \frac{2}{ab + bc + ca}. \]

1.4. Let $a$, $b$, $c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(b + c)}{a^2 + bc} + \frac{b(c + a)}{b^2 + ca} + \frac{c(a + b)}{c^2 + ab} \geq 2.$$
1.5. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \geq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}.
\]

1.6. Let \(a, b, c\) be positive real numbers. Prove that
\[
\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \geq \frac{a}{a^2 + bc} + \frac{b}{b^2 + ca} + \frac{c}{c^2 + ab}.
\]

1.7. Let \(a, b, c\) be positive real numbers. Prove that
\[
\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \geq \frac{2a}{3a^2 + bc} + \frac{2b}{3b^2 + ca} + \frac{2c}{3c^2 + ab}.
\]

1.8. Let \(a, b, c\) be positive real numbers. Prove that
\[
\begin{align*}
(a) & \quad \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{13}{6} - \frac{2(ab + bc + ca)}{3(a^2 + b^2 + c^2)}; \\
(b) & \quad \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} - \frac{3}{2} \geq (\sqrt{3} - 1) \left(1 - \frac{ab + bc + ca}{a^2 + b^2 + c^2}\right).
\end{align*}
\]

1.9. Let \(a, b, c\) be positive real numbers. Prove that
\[
\frac{1}{a^2 + 2bc} + \frac{1}{b^2 + 2ca} + \frac{1}{c^2 + 2ab} \leq \left(\frac{a + b + c}{ab + bc + ca}\right)^2.
\]

1.10. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2(b + c)}{b^2 + c^2} + \frac{b^2(c + a)}{c^2 + a^2} + \frac{c^2(a + b)}{a^2 + b^2} \geq a + b + c.
\]

1.11. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} \leq \frac{3(a^2 + b^2 + c^2)}{a + b + c}.
\]
1.12. Let $a, b, c$ be positive real numbers. Prove that
\[
\frac{1}{a^2 + ab + b^2} + \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \geq \frac{9}{(a + b + c)^2}.
\]

1.13. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2}{(2a + b)(2a + c)} + \frac{b^2}{(2b + c)(2b + a)} + \frac{c^2}{(2c + a)(2c + b)} \leq \frac{1}{3}.
\]

1.14. Let $a, b, c$ be positive real numbers. Prove that
\[
\begin{align*}
(a) & \quad \sum \frac{a}{(2a + b)(2a + c)} \leq \frac{1}{a + b + c}; \\
(b) & \quad \sum \frac{a^3}{(2a^2 + b^2)(2a^2 + c^2)} \leq \frac{1}{a + b + c}.
\end{align*}
\]

1.15. If $a, b, c$ are positive real numbers, then
\[
\sum \frac{1}{(a + 2b)(a + 2c)} \geq \frac{1}{(a + b + c)^2} + \frac{2}{3(a b + bc + ca)}.
\]

1.16. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that
\[
\begin{align*}
(a) & \quad \frac{1}{(a - b)^2} + \frac{1}{(b - c)^2} + \frac{1}{(c - a)^2} \geq \frac{4}{ab + bc + ca}; \\
(b) & \quad \frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \geq \frac{3}{ab + bc + ca}; \\
(c) & \quad \frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{5}{2(ab + bc + ca)}.
\end{align*}
\]

1.17. Let $a, b, c$ be positive real numbers, no two of which are zero. Prove that
\[
\frac{(a^2 + b^2)(a^2 + c^2)}{(a + b)(a + c)} + \frac{(b^2 + c^2)(b^2 + a^2)}{(b + c)(b + a)} + \frac{(c^2 + a^2)(c^2 + b^2)}{(c + a)(c + b)} \geq a^2 + b^2 + c^2.
\]
1.18. Let $a, b, c$ be positive real numbers such that $a + b + c = 3$. Prove that
\[
\frac{1}{a^2 + b + c} + \frac{1}{b^2 + c + a} + \frac{1}{c^2 + a + b} \leq 1.
\]

1.19. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that
\[
\frac{a^2 - bc}{a^2 + 3} + \frac{b^2 - ca}{b^2 + 3} + \frac{c^2 - ab}{c^2 + 3} \geq 0.
\]

1.20. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that
\[
\frac{1 - bc}{5 + 2a} + \frac{1 - ca}{5 + 2b} + \frac{1 - ab}{5 + 2c} \geq 0.
\]

1.21. Let $a, b, c$ be positive real numbers such that $a + b + c = 3$. Prove that
\[
\frac{1}{a^2 + b^2 + 2} + \frac{1}{b^2 + c^2 + 2} + \frac{1}{c^2 + a^2 + 2} \leq \frac{3}{4}.
\]

1.22. Let $a, b, c$ be positive real numbers such that $a + b + c = 3$. Prove that
\[
\frac{1}{4a^2 + b^2 + c^2} + \frac{1}{4b^2 + c^2 + a^2} + \frac{1}{4c^2 + a^2 + b^2} \leq \frac{1}{2}.
\]

1.23. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 2$. Prove that
\[
\frac{bc}{a^2 + 1} + \frac{ca}{b^2 + 1} + \frac{ab}{c^2 + 1} \leq 1.
\]

1.24. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 1$. Prove that
\[
\frac{bc}{a + 1} + \frac{ca}{b + 1} + \frac{ab}{c + 1} \leq \frac{1}{4}.
\]

1.25. Let $a, b, c$ be positive real numbers such that $a + b + c = 1$. Prove that
\[
\frac{1}{a(2a^2 + 1)} + \frac{1}{b(2b^2 + 1)} + \frac{1}{c(2c^2 + 1)} \leq \frac{3}{11abc}.
\]
1.26. Let $a, b, c$ be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^3 + b + c} + \frac{1}{b^3 + c + a} + \frac{1}{c^3 + a + b} \leq 1.$$ 

1.27. Let $a, b, c$ be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a^2}{1 + b^3 + c^3} + \frac{b^2}{1 + c^3 + a^3} + \frac{c^2}{1 + a^3 + b^3} \geq 1.$$ 

1.28. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{6 - ab} + \frac{1}{6 - bc} + \frac{1}{6 - ca} \leq \frac{3}{5}.$$ 

1.29. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{2a^2 + 7} + \frac{1}{2b^2 + 7} + \frac{1}{2c^2 + 7} \leq \frac{1}{3}.$$ 

1.30. Let $a, b, c$ be nonnegative real numbers such that $a \geq b \geq 1 \geq c$ and $a + b + c = 3$. Prove that

$$\frac{1}{a^2 + 3} + \frac{1}{b^2 + 3} + \frac{1}{c^2 + 3} \leq \frac{3}{4}.$$ 

1.31. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{2a^2 + 3} + \frac{1}{2b^2 + 3} + \frac{1}{2c^2 + 3} \geq \frac{3}{5}.$$ 

1.32. Let $a, b, c$ be nonnegative real numbers such that $a \geq 1 \geq b \geq c$ and $a + b + c = 3$. Prove that

$$\frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2} \geq 1.$$ 

1.33. Let $a, b, c$ be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{a + b + c}{6} + \frac{3}{a + b + c}.$$
1.34. Let \(a, b, c\) be nonnegative real numbers such that \(ab + bc + ca = 3\). Prove that

\[
\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} \geq \frac{3}{2}.
\]

1.35. Let \(a, b, c\) be positive real numbers such that \(ab + bc + ca = 3\). Prove that

\[
\frac{a^2}{a^2 + b + c} + \frac{b^2}{b^2 + c + a} + \frac{c^2}{c^2 + a + b} \geq 1.
\]

1.36. Let \(a, b, c\) be positive real numbers such that \(ab + bc + ca = 3\). Prove that

\[
\frac{bc + 4}{a^2 + 4} + \frac{ca + 4}{b^2 + 4} + \frac{ab + 4}{c^2 + 4} \leq 3 \leq \frac{bc + 2}{a^2 + 2} + \frac{ca + 2}{b^2 + 2} + \frac{ab + 2}{c^2 + 2}.
\]

1.37. Let \(a, b, c\) be nonnegative real numbers such that \(ab + bc + ca = 3\). If \(k \geq 2 + \sqrt{3}\), then

\[
\frac{1}{a + k} + \frac{1}{b + k} + \frac{1}{c + k} \leq \frac{3}{1 + k}.
\]

1.38. Let \(a, b, c\) be nonnegative real numbers such that \(a^2 + b^2 + c^2 = 3\). Prove that

\[
\frac{a(b + c)}{1 + bc} + \frac{b(c + a)}{1 + ca} + \frac{c(a + b)}{1 + ab} \leq 3.
\]

1.39. Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 = 3\). Prove that

\[
\frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} \leq 3.
\]

1.40. Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 = 3\). Prove that

\[
\frac{ab}{a + b} + \frac{bc}{b + c} + \frac{ca}{c + a} + 2 \leq \frac{7}{6}(a + b + c).
\]
1.41. Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 = 3\). Prove that

(a) \[
\frac{1}{3-ab} + \frac{1}{3-bc} + \frac{1}{3-ca} \leq \frac{3}{2};
\]

(b) \[
\frac{1}{\sqrt{6}-ab} + \frac{1}{\sqrt{6}-bc} + \frac{1}{\sqrt{6}-ca} \leq \frac{3}{\sqrt{6}-1}.
\]

1.42. Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 = 3\). Prove that

\[
\frac{1}{1+a^5} + \frac{1}{1+b^5} + \frac{1}{1+c^5} \geq \frac{3}{2}.
\]

1.43. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that

\[
\frac{1}{a^2+a+1} + \frac{1}{b^2+b+1} + \frac{1}{c^2+c+1} \geq 1.
\]

1.44. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that

\[
\frac{1}{a^2-a+1} + \frac{1}{b^2-b+1} + \frac{1}{c^2-c+1} \leq 3.
\]

1.45. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that

\[
\frac{3+a}{(1+a)^2} + \frac{3+b}{(1+b)^2} + \frac{3+c}{(1+c)^2} \geq 3.
\]

1.46. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that

\[
\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} \geq 1.
\]

1.47. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that

\[
\frac{a^6}{1+2a^5} + \frac{b^6}{1+2b^5} + \frac{c^6}{1+2c^5} \geq 1.
\]
1.48. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that
\[
\frac{a}{a^2 + 5} + \frac{b}{b^2 + 5} + \frac{c}{c^2 + 5} \leq \frac{1}{2}.
\]

1.49. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that
\[
\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} + \frac{1}{(1 + c)^2} + \frac{2}{(1 + a)(1 + b)(1 + c)} \geq 1.
\]

1.50. Let \(a, b, c\) be nonnegative real numbers such that
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} = \frac{3}{2}.
\]
Prove that
\[
\frac{3}{a + b + c} \geq \frac{2}{ab + bc + ca} + \frac{1}{a^2 + b^2 + c^2}.
\]

1.51. Let \(a, b, c\) be nonnegative real numbers such that
\[
7(a^2 + b^2 + c^2) = 11(ab + bc + ca).
\]
Prove that
\[
\frac{51}{28} \leq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leq 2.
\]

1.52. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{10}{(a + b + c)^2}.
\]

1.53. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \geq \frac{3}{\max\{ab, bc, ca\}}.
\]

1.54. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a(2a + b + c)}{b^2 + c^2} + \frac{b(2b + c + a)}{c^2 + a^2} + \frac{c(2c + a + b)}{a^2 + b^2} \geq 6.
\]
1.55. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2(b + c)^2}{b^2 + c^2} + \frac{b^2(c + a)^2}{c^2 + a^2} + \frac{c^2(a + b)^2}{a^2 + b^2} \geq 2(ab + bc + ca).
\]

1.56. If $a, b, c$ are real numbers such that $abc > 0$, then
\[
3 \sum \frac{a}{b^2 - bc + c^2} + 5 \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) \geq 8 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).
\]

1.57. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that
(a) \[2abc \left(\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a}\right) + a^2 + b^2 + c^2 \geq 2(ab + bc + ca);
\]
(b) \[
\frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + a} \leq \frac{3(a^2 + b^2 + c^2)}{2(a + b + c)}.
\]

1.58. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that
(a) \[
\frac{a^2 - bc}{b^2 + c^2} + \frac{b^2 - ca}{c^2 + a^2} + \frac{c^2 - ab}{a^2 + b^2} + \frac{3(ab + bc + ca)}{a^2 + b^2 + c^2} \geq 3;
\]
(b) \[
\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} + \frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq \frac{5}{2};
\]
(c) \[
\frac{a^2 + bc}{b^2 + c^2} + \frac{b^2 + ca}{c^2 + a^2} + \frac{c^2 + ab}{a^2 + b^2} \geq \frac{ab + bc + ca}{a^2 + b^2 + c^2} + 2.
\]

1.59. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \geq \frac{(a + b + c)^2}{2(ab + bc + ca)}.
\]

1.60. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{2ab}{(a + b)^2} + \frac{2bc}{(b + c)^2} + \frac{2ca}{(c + a)^2} + \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq \frac{5}{2}.
\]
1.61. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} + \frac{1}{4} \geq \frac{ab + bc + ca}{a^2 + b^2 + c^2}.
\]

1.62. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{3ab}{(a+b)^2} + \frac{3bc}{(b+c)^2} + \frac{3ca}{(c+a)^2} \leq \frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{5}{4}.
\]

1.63. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

(a) \[\frac{a^3 + abc}{b+c} + \frac{b^3 + abc}{c+a} + \frac{c^3 + abc}{a+b} \geq a^2 + b^2 + c^2;\]

(b) \[\frac{a^3 + 2abc}{b+c} + \frac{b^3 + 2abc}{c+a} + \frac{c^3 + 2abc}{a+b} \geq \frac{1}{2}(a+b+c)^2;\]

(c) \[\frac{a^3 + 3abc}{b+c} + \frac{b^3 + 3abc}{c+a} + \frac{c^3 + 3abc}{a+b} \geq 2(ab + bc + ca).\]

1.64. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

\[\frac{a^2 + 3abc}{(b+c)^2} + \frac{b^2 + 3abc}{(c+a)^2} + \frac{c^2 + 3abc}{(a+b)^2} \geq a + b + c.\]

1.65. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

(a) \[\frac{a^3 + 3abc}{(b+c)^3} + \frac{b^3 + 3abc}{(c+a)^3} + \frac{c^3 + 3abc}{(a+b)^3} \geq \frac{3}{2};\]

(b) \[\frac{3a^3 + 13abc}{(b+c)^3} + \frac{3b^3 + 13abc}{(c+a)^3} + \frac{3c^3 + 13abc}{(a+b)^3} \geq 6.\]

1.66. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

(a) \[\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} + ab + bc + ca \geq \frac{3}{2}(a^2 + b^2 + c^2);\]

(b) \[\frac{2a^2 + bc}{b+c} + \frac{2b^2 + ca}{c+a} + \frac{2c^2 + ab}{a+b} \geq \frac{9(a^2 + b^2 + c^2)}{2(a+b+c)}.\]
1.67. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a(b + c)}{b^2 + bc + c^2} + \frac{b(c + a)}{c^2 + ca + a^2} + \frac{c(a + b)}{a^2 + ab + b^2} \geq 2.
\]

1.68. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a(b + c)}{b^2 + bc + c^2} + \frac{b(c + a)}{c^2 + ca + a^2} + \frac{c(a + b)}{a^2 + ab + b^2} \geq 2 + 4 \prod \left( \frac{a - b}{a + b} \right)^2.
\]

1.69. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{ab - bc + ca}{b^2 + c^2} + \frac{bc - ca + ab}{c^2 + a^2} + \frac{ca - ab + bc}{a^2 + b^2} \geq \frac{3}{2}.
\]

1.70. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If \( k > -2 \), then
\[
\sum \frac{ab + (k - 1)bc + ca}{b^2 + kbc + c^2} \geq \frac{3(k + 1)}{k + 2}.
\]

1.71. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If \( k > -2 \), then
\[
\sum \frac{3bc - a(b + c)}{b^2 + kbc + c^2} \leq \frac{3}{k + 2}.
\]

1.72. Let \( a, b, c \) be nonnegative real numbers such that \( ab + bc + ca = 3 \). Prove that
\[
\frac{ab + 1}{a^2 + b^2} + \frac{bc + 1}{b^2 + c^2} + \frac{ca + 1}{c^2 + a^2} \geq 4 \cdot \frac{3}{3}.
\]

1.73. Let \( a, b, c \) be nonnegative real numbers such that \( ab + bc + ca = 3 \). Prove that
\[
\frac{5ab + 1}{(a + b)^2} + \frac{5bc + 1}{(b + c)^2} + \frac{5ca + 1}{(c + a)^2} \geq 2.
\]

1.74. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2 - bc}{2b^2 - 3bc + 2c^2} + \frac{b^2 - ca}{2c^2 - 3ca + 2a^2} + \frac{c^2 - ab}{2a^2 - 3ab + 2b^2} \geq 0.
\]
1.75. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2 - bc}{b^2 - bc + c^2} + \frac{2b^2 - ca}{c^2 - ca + a^2} + \frac{2c^2 - ab}{a^2 - ab + b^2} \geq 3.$$  

1.76. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{2b^2 - bc + 2c^2} + \frac{b^2}{2c^2 - ca + 2a^2} + \frac{c^2}{2a^2 - ab + 2b^2} \geq 1.$$  

1.77. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{4b^2 - bc + 4c^2} + \frac{1}{4c^2 - ca + 4a^2} + \frac{1}{4a^2 - ab + 4b^2} \geq \frac{9}{7(a^2 + b^2 + c^2)}.$$  

1.78. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2 + bc}{b^2 + c^2} + \frac{2b^2 + ca}{c^2 + a^2} + \frac{2c^2 + ab}{a^2 + b^2} \geq \frac{9}{2}.$$  

1.79. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2 + 3bc}{b^2 + bc + c^2} + \frac{2b^2 + 3ca}{c^2 + ca + a^2} + \frac{2c^2 + 3ab}{a^2 + ab + b^2} \geq 5.$$  

1.80. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2 + 5bc}{(b + c)^2} + \frac{2b^2 + 5ca}{(c + a)^2} + \frac{2c^2 + 5ab}{(a + b)^2} \geq \frac{21}{4}.$$  

1.81. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. If $k > -2$, then

$$\sum \frac{2a^2 + (2k + 1)bc}{b^2 + kbc + c^2} \geq \frac{3(2k + 3)}{k + 2}.$$
1.82. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(k > -2\), then
\[
\sum \frac{3bc - 2a^2}{b^2 + kbc + c^2} \leq \frac{3}{k + 2}.
\]

1.83. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{a^2 + 16bc}{b^2 + c^2} + \frac{b^2 + 16ca}{c^2 + a^2} + \frac{c^2 + 16ab}{a^2 + b^2} \geq 10.
\]

1.84. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{a^2 + 128bc}{b^2 + c^2} + \frac{b^2 + 128ca}{c^2 + a^2} + \frac{c^2 + 128ab}{a^2 + b^2} \geq 46.
\]

1.85. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{a^2 + 64bc}{(b + c)^2} + \frac{b^2 + 64ca}{(c + a)^2} + \frac{c^2 + 64ab}{(a + b)^2} \geq 18.
\]

1.86. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(k \geq -1\), then
\[
\sum \frac{a^2(b + c) + kabc}{b^2 + kbc + c^2} \geq a + b + c.
\]

1.87. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(k \geq -\frac{3}{2}\), then
\[
\sum \frac{a^3 + (k + 1)abc}{b^2 + kbc + c^2} \geq a + b + c.
\]

1.88. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(k > 0\), then
\[
\frac{2a^k - b^k - c^k}{b^2 - bc + c^2} + \frac{2b^k - c^k - a^k}{c^2 - ca + a^2} + \frac{2c^k - a^k - b^k}{a^2 - ab + b^2} \geq 0.
\]
1.89. If $a$, $b$, $c$ are the lengths of the sides of a triangle, then

(a) $\frac{b + c - a}{b^2 - bc + c^2} + \frac{c + a - b}{c^2 - ca + a^2} + \frac{a + b - c}{a^2 - ab + b^2} \leq \frac{2(a + b + c)}{a^2 + b^2 + c^2}$;

(b) $\frac{a^2 - 2bc}{b^2 - bc + c^2} + \frac{b^2 - 2ca}{c^2 - ca + a^2} + \frac{c^2 - 2ab}{a^2 - ab + b^2} \leq 0$.

1.90. If $a$, $b$, $c$ are nonnegative real numbers, then

$$\frac{a^2}{5a^2 + (b + c)^2} + \frac{b^2}{5b^2 + (c + a)^2} + \frac{c^2}{5c^2 + (a + b)^2} \leq \frac{1}{3}.$$  

1.91. If $a$, $b$, $c$ are nonnegative real numbers, then

$$\frac{b^2 + c^2 - a^2}{2a^2 + (b + c)^2} + \frac{c^2 + a^2 - b^2}{2b^2 + (c + a)^2} + \frac{a^2 + b^2 - c^2}{2c^2 + (a + b)^2} \geq \frac{1}{2}.$$  

1.92. Let $a$, $b$, $c$ be positive real numbers. If $k > 0$, then

$$\frac{3a^2 - 2bc}{ka^2 + (b - c)^2} + \frac{3b^2 - 2ca}{kb^2 + (c - a)^2} + \frac{3c^2 - 2ab}{kc^2 + (a - b)^2} \leq \frac{3}{k}.$$  

1.93. Let $a$, $b$, $c$ be nonnegative real numbers, no two of which are zero. If $k \geq 3 + \sqrt{7}$, then

(a) $\frac{a}{a^2 + kbc} + \frac{b}{b^2 + kca} + \frac{c}{c^2 + kab} \geq \frac{9}{(1 + k)(a + b + c)}$;

(b) $\frac{1}{ka^2 + bc} + \frac{1}{kb^2 + ca} + \frac{1}{kc^2 + ab} \geq \frac{9}{(k + 1)(ab + bc + ca)}$.

1.94. Let $a$, $b$, $c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq \frac{6}{a^2 + b^2 + c^2 + ab + bc + ca}.$$
1.95. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{2a^2 + 5bc} + \frac{1}{2b^2 + 5ca} + \frac{1}{2c^2 + 5ab} \geq \frac{1}{(a + b + c)^2}.
\]

1.96. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq \frac{8}{(a + b + c)^2}.
\]

1.97. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{a^2 + bc} + \frac{1}{b^2 + ca} + \frac{1}{c^2 + ab} \geq \frac{12}{(a + b + c)^2}.
\]

1.98. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\begin{align*}
(a) & \quad \frac{1}{a^2 + 2bc} + \frac{1}{b^2 + 2ca} + \frac{1}{c^2 + 2ab} \geq \frac{1}{a^2 + b^2 + c^2} + \frac{2}{ab + bc + ca}; \\
(b) & \quad \frac{a(b + c)}{a^2 + 2bc} + \frac{b(c + a)}{b^2 + 2ca} + \frac{c(a + b)}{c^2 + 2ab} \geq 1 + \frac{ab + bc + ca}{a^2 + b^2 + c^2}.
\end{align*}
\]

1.99. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\begin{align*}
(a) & \quad \frac{a}{a^2 + 2bc} + \frac{b}{b^2 + 2ca} + \frac{c}{c^2 + 2ab} \leq \frac{a + b + c}{ab + bc + ca}; \\
(b) & \quad \frac{a(b + c)}{a^2 + 2bc} + \frac{b(c + a)}{b^2 + 2ca} + \frac{c(a + b)}{c^2 + 2ab} \leq 1 + \frac{a^2 + b^2 + c^2}{ab + bc + ca}.
\end{align*}
\]

1.100. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\begin{align*}
(a) & \quad \frac{a}{2a^2 + bc} + \frac{b}{2b^2 + ca} + \frac{c}{2c^2 + ab} \geq \frac{a + b + c}{a^2 + b^2 + c^2}; \\
(b) & \quad \frac{b + c}{2a^2 + bc} + \frac{c + a}{2b^2 + ca} + \frac{a + b}{2c^2 + ab} \geq \frac{6}{a + b + c}.
\end{align*}
\]
1.101. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{a(b + c)}{a^2 + bc} + \frac{b(c + a)}{b^2 + ca} + \frac{c(a + b)}{c^2 + ab} \geq \frac{(a + b + c)^2}{a^2 + b^2 + c^2}.
\]

1.102. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If \( k > 0 \), then

\[
\frac{b^2 + c^2 + \sqrt{3}bc}{a^2 + kbc} + \frac{c^2 + a^2 + \sqrt{3}ca}{b^2 + kca} + \frac{a^2 + b^2 + \sqrt{3}ab}{c^2 + kab} \geq \frac{3(2 + \sqrt{3})}{1 + k}.
\]

1.103. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{8}{a^2 + b^2 + c^2} \geq \frac{6}{ab + bc + ca}.
\]

1.104. If \( a, b, c \) are the lengths of the sides of a triangle, then

\[
\frac{a(b + c)}{a^2 + 2bc} + \frac{b(c + a)}{b^2 + 2ca} + \frac{c(a + b)}{c^2 + 2ab} \leq 2.
\]

1.105. If \( a, b, c \) are real numbers, then

\[
\frac{a^2 - bc}{2a^2 + b^2 + c^2} + \frac{b^2 - ca}{2b^2 + c^2 + a^2} + \frac{c^2 - ab}{2c^2 + a^2 + b^2} \geq 0.
\]

1.106. If \( a, b, c \) are nonnegative real numbers, then

\[
\frac{3a^2 - bc}{2a^2 + b^2 + c^2} + \frac{3b^2 - ca}{2b^2 + c^2 + a^2} + \frac{3c^2 - ab}{2c^2 + a^2 + b^2} \leq \frac{3}{2}.
\]

1.107. If \( a, b, c \) are nonnegative real numbers, then

\[
\frac{(b + c)^2}{4a^2 + b^2 + c^2} + \frac{(c + a)^2}{4b^2 + c^2 + a^2} + \frac{(a + b)^2}{4c^2 + a^2 + b^2} \geq 2.
\]
1.108. If \(a, b, c\) are positive real numbers, then

(a) \[\sum \frac{1}{11a^2 + 2b^2 + 2c^2} \leq \frac{3}{5(ab + bc + ca)};\]

(b) \[\sum \frac{1}{4a^2 + b^2 + c^2} \leq \frac{1}{2(a^2 + b^2 + c^2)} + \frac{1}{ab + bc + ca}.\]

1.109. If \(a, b, c\) are nonnegative real numbers such that \(ab + bc + ca = 3\), then

\[\frac{\sqrt{a}}{b + c} + \frac{\sqrt{b}}{c + a} + \frac{\sqrt{c}}{a + b} \geq \frac{3}{2}.\]

1.110. If \(a, b, c\) are nonnegative real numbers such that \(ab + bc + ca \geq 3\), then

\[\frac{1}{2 + a} + \frac{1}{2 + b} + \frac{1}{2 + c} \geq \frac{1}{1 + b + c} + \frac{1}{1 + c + a} + \frac{1}{1 + a + b}.\]

1.111. If \(a, b, c\) are the lengths of the sides of a triangle, then

(a) \[\frac{a^2 - bc}{3a^2 + b^2 + c^2} + \frac{b^2 - ca}{3b^2 + c^2 + a^2} + \frac{c^2 - ab}{3c^2 + a^2 + b^2} \leq 0;\]

(b) \[\frac{a^4 - b^2c^2}{3a^4 + b^4 + c^4} + \frac{b^4 - c^2a^2}{3b^4 + c^4 + a^4} + \frac{c^4 - a^2b^2}{3c^4 + a^4 + b^4} \leq 0.\]

1.112. If \(a, b, c\) are the lengths of the sides of a triangle, then

\[\frac{bc}{4a^2 + b^2 + c^2} + \frac{ca}{4b^2 + c^2 + a^2} + \frac{ab}{4c^2 + a^2 + b^2} \geq \frac{1}{2}.\]

1.113. If \(a, b, c\) are the lengths of the sides of a triangle, then

\[\frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{1}{a^2 + b^2} \leq \frac{9}{2(ab + bc + ca)}.\]
1.114. If \(a, b, c\) are the lengths of the sides of a triangle, then

(a) \[\frac{a + b}{a - b} + \frac{b + c}{b - c} + \frac{c + a}{c - a} \geq 5;\]

(b) \[\frac{a^2 + b^2}{a^2 - b^2} + \frac{b^2 + c^2}{b^2 - c^2} + \frac{c^2 + a^2}{c^2 - a^2} \geq 3.\]

1.115. If \(a, b, c\) are the lengths of the sides of a triangle, then

\[\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} + 3 \geq 6 \left(\frac{a}{a + b} + \frac{b}{b + c} + \frac{c}{c + a}\right).\]

1.116. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

\[\sum\frac{3a(b + c) - 2bc}{(b + c)(2a + b + c)} \geq \frac{3}{2}.\]

1.117. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

\[\sum\frac{a(b + c) - 2bc}{(b + c)(3a + b + c)} \geq 0.\]

1.118. Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 \geq 3\). Prove that

\[\frac{a^5 - a^2}{a^5 + b^2 + c^2} + \frac{b^5 - b^2}{b^5 + c^2 + a^2} + \frac{c^5 - c^2}{c^5 + a^2 + b^2} \geq 0.\]

1.119. Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 = a^3 + b^3 + c^3\). Prove that

\[\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b} \geq \frac{3}{2}.\]

1.120. If \(a, b, c \in [0, 1]\), then

(a) \[\frac{a}{bc + 2} + \frac{b}{ca + 2} + \frac{c}{ab + 2} \leq 1;\]

(b) \[\frac{ab}{2bc + 1} + \frac{bc}{2ca + 1} + \frac{ca}{2ab + 1} \leq 1.\]
1.121. Let $a, b, c$ be positive real numbers such that $a + b + c = 2$. Prove that
\[
5(1 - ab - bc - ca)\left(\frac{1}{1 - ab} + \frac{1}{1 - bc} + \frac{1}{1 - ca}\right) + 9 \geq 0.
\]

1.122. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 2$. Prove that
\[
2 - a^2 \leq 2 - bc \leq 3.
\]

1.123. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that
\[
\frac{3 + 5a^2}{3 - bc} + \frac{3 + 5b^2}{3 - ca} + \frac{3 + 5c^2}{3 - ab} \geq 12.
\]

1.124. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 2$. If \[-\frac{1}{7} \leq m \leq \frac{7}{8},\] then
\[
\frac{a^2 + m}{3 - 2bc} + \frac{b^2 + m}{3 - 2ca} + \frac{c^2 + m}{3 - 2ab} \geq \frac{3(4 + 9m)}{19}.
\]

1.125. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that
\[
\frac{47 - 7a^2}{1 + bc} + \frac{47 - 7b^2}{1 + ca} + \frac{47 - 7c^2}{1 + ab} \geq 60.
\]

1.126. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that
\[
\frac{26 - 7a^2}{1 + bc} + \frac{26 - 7b^2}{1 + ca} + \frac{26 - 7c^2}{1 + ab} \leq \frac{57}{2}.
\]

1.127. Let $a, b, c$ be nonnegative real numbers, no all are zero. Prove that
\[
\sum \frac{5a(b + c) - 6bc}{a^2 + b^2 + c^2 + bc} \leq 3.
\]
Let \( a, b, c \) be nonnegative real numbers, no two of which are zero, and let 
\[
x = \frac{a^2 + b^2 + c^2}{ab + bc + ca}.
\]
Prove that
(a) \( \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} + \frac{1}{2} \geq x + \frac{1}{x} \);
(b) \( 6 \left( \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \right) \geq 5x + \frac{4}{x} \);
(c) \( \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} - \frac{3}{2} \geq \frac{1}{3} \left( x - \frac{1}{x} \right) \).

If \( a, b, c \) are real numbers, then
\[
\frac{1}{a^2 + 7(b^2 + c^2)} + \frac{1}{b^2 + 7(c^2 + a^2)} + \frac{1}{c^2 + 7(a^2 + b^2)} \leq \frac{9}{5(a + b + c)^2}.
\]

If \( a, b, c \) are real numbers, then
\[
\frac{bc}{3a^2 + b^2 + c^2} + \frac{ca}{3b^2 + c^2 + a^2} + \frac{ab}{3c^2 + a^2 + b^2} \leq \frac{3}{5}.
\]

If \( a, b, c \) are real numbers such that \( a + b + c = 3 \), then
(a) \( \frac{1}{2 + b^2 + c^2} + \frac{1}{2 + c^2 + a^2} + \frac{1}{2 + a^2 + b^2} \leq \frac{3}{4} \);
(b) \( \frac{1}{8 + 5(b^2 + c^2)} + \frac{1}{8 + 5(c^2 + a^2)} + \frac{1}{8 + 5(a^2 + b^2)} \leq \frac{1}{6} \).

If \( a, b, c \) are real numbers, then
\[
\frac{(a + b)(a + c)}{a^2 + 4(b^2 + c^2)} + \frac{(b + c)(b + a)}{b^2 + 4(c^2 + a^2)} + \frac{(c + a)(c + b)}{c^2 + 4(a^2 + b^2)} \leq \frac{4}{3}.
\]
1.133. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\sum \frac{1}{(b + c)(7a + b + c)} \leq \frac{1}{2(ab + bc + ca)}.
\]

1.134. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\sum \frac{1}{b^2 + c^2 + 4a(b + c)} \leq \frac{9}{10(ab + bc + ca)}.
\]

1.135. If \(a, b, c\) are nonnegative real numbers such that \(a + b + c = 3\), then
\[
\frac{1}{3 - ab} + \frac{1}{3 - bc} + \frac{1}{3 - ca} \leq \frac{9}{2(ab + bc + ca)}.
\]

1.136. If \(a, b, c\) are nonnegative real numbers such that \(a + b + c = 3\), then
\[
\frac{bc}{a^2 + a + 6} + \frac{ca}{b^2 + b + 6} + \frac{ab}{c^2 + c + 6} \leq \frac{3}{8}.
\]

1.137. If \(a, b, c\) are nonnegative real numbers such that \(ab + bc + ca = 3\), then
\[
\frac{1}{8a^2 - 2bc + 21} + \frac{1}{8b^2 - 2ca + 21} + \frac{1}{8c^2 - 2ab + 21} \geq \frac{1}{9}.
\]

1.138. Let \(a, b, c\) be real numbers, no two of which are zero. Prove that
\[
\begin{align*}
\text{(a)} & \quad \frac{a^2 + bc}{b^2 + c^2} + \frac{b^2 + ca}{c^2 + a^2} + \frac{c^2 + ab}{a^2 + b^2} \geq \frac{(a + b + c)^2}{a^2 + b^2 + c^2}; \\
\text{(b)} & \quad \frac{a^2 + 3bc}{b^2 + c^2} + \frac{b^2 + 3ca}{c^2 + a^2} + \frac{c^2 + 3ab}{a^2 + b^2} \geq \frac{6(ab + bc + ca)}{a^2 + b^2 + c^2}.
\end{align*}
\]

1.139. Let \(a, b, c\) be real numbers such that \(ab + bc + ca \geq 0\) and no two of which are zero. Prove that
\[
\frac{a(b + c)}{b^2 + c^2} + \frac{b(c + a)}{c^2 + a^2} + \frac{c(a + b)}{a^2 + b^2} \geq \frac{3}{10}.
\]
1.140. If \(a, b, c\) are positive real numbers such that \(abc > 1\), then
\[
\frac{1}{a + b + c - 3} + \frac{1}{abc - 1} \geq \frac{4}{ab + bc + ca - 3}.
\]

1.141. Let \(a, b, c\) be positive real numbers, no two of which are zero. Prove that
\[
\sum \left(\frac{4b^2 - ac}{b + c}\right) \leq \frac{27}{2} abc.
\]

1.142. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero, such that \(a + b + c = 3\).

Prove that
\[
\frac{a}{3a + bc} + \frac{b}{3b + ca} + \frac{c}{3c + ab} \geq \frac{2}{3}.
\]

1.143. Let \(a, b, c\) be positive real numbers such that
\[
(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 10.
\]

Prove that
\[
\frac{19}{12} \leq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leq \frac{5}{3}.
\]

1.144. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero, such that \(a + b + c = 3\). Prove that
\[
\frac{9}{10} < \frac{a}{2a + bc} + \frac{b}{2b + ca} + \frac{c}{2c + ab} \leq 1.
\]

1.145. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^3}{2a^2 + bc} + \frac{b^3}{2b^2 + ca} + \frac{c^3}{2c^2 + ab} \leq \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}.
\]
1.146. Let \(a, b, c\) be positive real numbers, no two of which are zero. Prove that
\[
\frac{a^3}{4a^2 + bc} + \frac{b^3}{4b^2 + ca} + \frac{c^3}{4c^2 + ab} \geq \frac{a + b + c}{5}.
\]

1.147. If \(a, b, c\) are positive real numbers, then
\[
\frac{1}{(2 + a)^2} + \frac{1}{(2 + b)^2} + \frac{1}{(2 + c)^2} \geq \frac{3}{6 + ab + bc + ca}.
\]

1.148. If \(a, b, c\) are positive real numbers, then
\[
\frac{1}{1 + 3a} + \frac{1}{1 + 3b} + \frac{1}{1 + 3c} \geq \frac{3}{3 + abc}.
\]

1.149. Let \(a, b, c\) be real numbers, no two of which are zero. If \(1 \leq k \leq 3\), then
\[
\left(k + \frac{2ab}{a^2 + b^2}\right)\left(k + \frac{2bc}{b^2 + c^2}\right)\left(k + \frac{2ca}{c^2 + a^2}\right) \geq (k-1)(k^2 - 1).
\]

1.150. If \(a, b, c\) are non-zero and distinct real numbers, then
\[
\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + 3\left[\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2}\right] \geq 4\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right).
\]

1.151. Let \(a, b, c\) be positive real numbers, and let
\[
A = \frac{a}{b} + \frac{b}{a} + k, \quad B = \frac{b}{c} + \frac{c}{b} + k, \quad C = \frac{c}{a} + \frac{a}{c} + k,
\]
where \(-2 < k \leq 4\). Prove that
\[
\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq \frac{1}{k+2} + \frac{4}{A+B+C-(k+2)}.
\]

1.152. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} \geq \frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab}.
\]
1.153. If $a, b, c$ are nonnegative real numbers such that $a + b + c = 3$, then
\[ \frac{1}{2ab + 1} + \frac{1}{2bc + 1} + \frac{1}{2ca + 1} \geq \frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2}. \]

1.154. If $a, b, c$ are nonnegative real numbers such that $a + b + c = 4$, then
\[ \frac{1}{ab + 2} + \frac{1}{bc + 2} + \frac{1}{ca + 2} \geq \frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2}. \]

1.155. If $a, b, c$ are nonnegative real numbers, no two of which are zero, then
(a) \[ \frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{(a - b)^2(b - c)^2(c - a)^2}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} \leq 1; \]
(b) \[ \frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{(a - b)^2(b - c)^2(c - a)^2}{(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2)} \leq 1. \]

1.156. If $a, b, c$ are nonnegative real numbers, no two of which are zero, then
\[ \frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{45}{8(a^2 + b^2 + c^2) + 2(ab + bc + ca)}. \]

1.157. If $a, b, c$ are real numbers, no two of which are zero, then
\[ \frac{a^2 - 7bc}{b^2 + c^2} + \frac{b^2 - 7ca}{a^2 + b^2} + \frac{c^2 - 7ab}{a^2 + b^2} + \frac{9(ab + bc + ca)}{a^2 + b^2 + c^2} \geq 0. \]

1.158. If $a, b, c$ are real numbers such that $abc \neq 0$, then
\[ \frac{(b + c)^2}{a^2} + \frac{(c + a)^2}{b^2} + \frac{(a + b)^2}{c^2} \geq 2 + \frac{10(a + b + c)^2}{3(a^2 + b^2 + c^2)}. \]

1.159. If $a, b, c$ are nonnegative real numbers, no two of which are zero, then
\[ \frac{a^2 - 4bc}{b^2 + c^2} + \frac{b^2 - 4ca}{a^2 + b^2} + \frac{c^2 - 4ab}{a^2 + b^2} + \frac{9(ab + bc + ca)}{a^2 + b^2 + c^2} \geq \frac{9}{2}. \]
1.160. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq 1 + \frac{9(a - b)^2(b - c)^2(c - a)^2}{(a + b)^2(b + c)^2(c + a)^2}.
\]

1.161. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq 1 + \frac{(a - b)^2(b - c)^2(c - a)^2}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}.
\]

1.162. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \geq \frac{5}{3a + b + c} + \frac{5}{3b + c + a} + \frac{5}{3c + a + b}.
\]

1.163. If \(a, b, c\) are real numbers, no two of which are zero, then
\[
\begin{align*}
(a) \quad & \quad \frac{8a^2 + 3bc}{b^2 + bc + c^2} + \frac{8b^2 + 3ca}{c^2 + ca + a^2} + \frac{8c^2 + 3ab}{a^2 + ab + b^2} \geq 11; \\
(b) \quad & \quad \frac{8a^2 - 5bc}{b^2 - bc + c^2} + \frac{8b^2 - 5ca}{c^2 - ca + a^2} + \frac{8c^2 - 5ab}{a^2 - ab + b^2} \geq 9.
\end{align*}
\]

1.164. If \(a, b, c\) are real numbers, no two of which are zero, then
\[
\frac{4a^2 + bc}{4b^2 + 7bc + 4c^2} + \frac{4b^2 + ca}{4c^2 + 7ca + 4a^2} + \frac{4c^2 + ab}{4a^2 + 7ab + 4b^2} \geq 1.
\]

1.165. If \(a, b, c\) are real numbers, no two of which are equal, then
\[
\frac{1}{(a - b)^2} + \frac{1}{(b - c)^2} + \frac{1}{(c - a)^2} \geq \frac{27}{4(a^2 + b^2 + c^2 - ab - bc - ca)}.
\]

1.166. If \(a, b, c\) are real numbers, no two of which are zero, then
\[
\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \geq \frac{14}{3(a^2 + b^2 + c^2)}.
\]
1.167. Let \( a, b, c \) be real numbers such that \( ab + bc + ca \geq 0 \) and no two of which are zero. Prove that

(a) \[
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2};
\]

(b) if \( ab \leq 0 \), then

\[
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2.
\]

1.168. If \( a, b, c \) are nonnegative real numbers, then

\[
\frac{a}{7a+b+c} + \frac{b}{7b+c+a} + \frac{c}{7c+a+b} \geq \frac{ab + bc + ca}{(a+b+c)^2}.
\]

1.169. If \( a, b, c \) are the lengths of the sides of a triangle, then

\[
\frac{a^2}{4a^2 + 5bc} + \frac{b^2}{4b^2 + 5ca} + \frac{c^2}{4c^2 + 5ab} \geq \frac{1}{3}.
\]

1.170. If \( a, b, c \) are the lengths of the sides of a triangle, then

\[
\frac{1}{7a^2 + b^2 + c^2} + \frac{1}{7b^2 + c^2 + a^2} + \frac{1}{7c^2 + a^2 + b^2} \geq \frac{3}{(a+b+c)^2}.
\]

1.171. Let \( a, b, c \) be the lengths of the sides of a triangle. If \( k > -2 \), then

\[
\sum \frac{a(b+c) + (k+1)bc}{b^2 + kbc + c^2} \leq \frac{3(k+3)}{k+2}.
\]

1.172. Let \( a, b, c \) be the lengths of the sides of a triangle. If \( k > -2 \), then

\[
\sum \frac{2a^2 + (4k + 9)bc}{b^2 + kbc + c^2} \leq \frac{3(4k+11)}{k+2}.
\]

1.173. If \( a \geq b \geq c \geq d \) such that \( abcd = 1 \), then

\[
\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{3}{1+\sqrt[3]{abc}}.
\]
1.174. Let $a, b, c, d$ be positive real numbers such that $abcd = 1$. Prove that

$$\sum \frac{1}{1 + ab + bc + ca} \leq 1.$$ 

1.175. Let $a, b, c, d$ be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1.$$ 

1.176. Let $a, b, c, d \neq \frac{1}{3}$ be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{(3a-1)^2} + \frac{1}{(3b-1)^2} + \frac{1}{(3c-1)^2} + \frac{1}{(3d-1)^2} \geq 1.$$ 

1.177. Let $a, b, c, d$ be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{1 + a + a^2 + a^3} + \frac{1}{1 + b + b^2 + b^3} + \frac{1}{1 + c + c^2 + c^3} + \frac{1}{1 + d + d^2 + d^3} \geq 1.$$ 

1.178. Let $a, b, c, d$ be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{1 + a + 2a^2} + \frac{1}{1 + b + 2b^2} + \frac{1}{1 + c + 2c^2} + \frac{1}{1 + d + 2d^2} \geq 1.$$ 

1.179. Let $a, b, c, d$ be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{9}{a + b + c + d} \geq \frac{25}{4}.$$ 

1.180. If $a, b, c, d$ are real numbers such that $a + b + c + d = 0$, then

$$\frac{(a-1)^2}{3a^2 + 1} + \frac{(b-1)^2}{3b^2 + 1} + \frac{(c-1)^2}{3c^2 + 1} + \frac{(d-1)^2}{3d^2 + 1} \leq 4.$$
1.181. If \(a, b, c, d \geq -5\) such that \(a + b + c + d = 4\), then
\[
\frac{1 - a}{(1 + a)^2} + \frac{1 - b}{(1 + b)^2} + \frac{1 - c}{(1 + c)^2} + \frac{1 - d}{(1 + d)^2} \geq 0.
\]

1.182. Let \(a_1, a_2, \ldots, a_n\) be positive real numbers such that \(a_1 + a_2 + \cdots + a_n = n\). Prove that
\[
\sum \frac{1}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} \leq \frac{1}{2}.
\]

1.183. Let \(a_1, a_2, \ldots, a_n\) be real numbers such that \(a_1 + a_2 + \cdots + a_n = 0\). Prove that
\[
\frac{(a_1 + 1)^2}{a_1^2 + n - 1} + \frac{(a_2 + 1)^2}{a_2^2 + n - 1} + \cdots + \frac{(a_n + 1)^2}{a_n^2 + n - 1} \geq \frac{n}{n-1}.
\]

1.184. Let \(a_1, a_2, \ldots, a_n\) be positive real numbers such that \(a_1a_2 \cdots a_n = 1\). Prove that
\[
\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} \geq 1.
\]

1.185. Let \(a_1, a_2, \ldots, a_n\) be positive real numbers such that \(a_1a_2 \cdots a_n = 1\). Prove that
\[
\frac{1}{1 - a_1 + na_1^2} + \frac{1}{1 - a_2 + na_2^2} + \cdots + \frac{1}{1 - a_n + na_n^2} \geq 1.
\]

1.186. Let \(a_1, a_2, \ldots, a_n\) be positive real numbers such that
\[
a_1, a_2, \ldots, a_n \geq \frac{k(n-k-1)}{kn-k-1}, \quad k > 1
\]
and
\[
a_1a_2 \cdots a_n = 1.
\]
Prove that
\[
\frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k} \leq \frac{n}{1 + k}.
\]
1.187. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that
\[ a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \quad a_1a_2\cdots a_n = 1. \]
Prove that
\[ \frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2} \geq 0. \]

1.188. If $a_1, a_2, \ldots, a_n \geq 0$, then
\[ \frac{1}{1 + na_1} + \frac{1}{1 + na_2} + \cdots + \frac{1}{1 + na_n} \geq \frac{n}{n + a_1a_2\cdots a_n}. \]

1.189. If $a_1, a_2, \ldots, a_n$ are positive real numbers, then
\[ \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \geq \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}, \]
where
\[ b_i = \frac{1}{n-1} \sum_{j \neq i} a_j, \quad i = 1, 2, \ldots, n. \]

1.190. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that
\[ a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}, \]
then
\begin{align*}
(a) \quad & \frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} \geq 1; \\
(b) \quad & \frac{1}{n-1 + a_1} + \frac{1}{n-1 + a_2} + \cdots + \frac{1}{n-1 + a_n} \leq 1.
\end{align*}
1.2 Solutions

P 1.1. If $a, b, c$ are nonnegative real numbers, then
\[
\frac{a^2 - bc}{3a + b + c} + \frac{b^2 - ca}{3b + c + a} + \frac{c^2 - ab}{3c + a + b} \geq 0.
\]

Solution. We use the SOS method. Without loss of generality, assume that $a \geq b \geq c$. We have
\[
2 \sum \frac{a^2 - bc}{3a + b + c} = \sum \frac{(a - b)(a + c) + (a - c)(a + b)}{3a + b + c}
\]
\[
= \sum \frac{(a - b)(a + c)}{3a + b + c} + \sum \frac{(b - a)(b + c)}{3b + c + a}
\]
\[
= \sum \frac{(a - b)^2(a + b - c)}{(3a + b + c)(3b + c + a)}
\]
Since $a + b - c \geq 0$, it suffices to show that
\[
(b - c)^2(b + c - a)(3a + b + c) + (c - a)^2(c + a - b)(3b + c + a) \geq 0;
\]
that is,
\[
(a - c)^2(c + a - b)(3b + c + a) \geq (b - c)^2(a - b - c)(3a + b + c).
\]
This inequality is trivial for $a \leq b + c$. Otherwise, we can get it by multiplying the obvious inequalities
\[
c + a - b \geq a - b - c,
\]
\[
b^2(a - c)^2 \geq a^2(b - c)^2,
\]
\[
a(3b + c + a) \geq b(3a + b + c),
\]
a \geq b.

The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

P 1.2. If $a, b, c$ are positive real numbers, then
\[
\frac{4a^2 - b^2 - c^2}{a(b + c)} + \frac{4b^2 - c^2 - a^2}{b(c + a)} + \frac{4c^2 - a^2 - b^2}{c(a + b)} \leq 3.
\]

(Vasile Cîrtoaje, 2006)
Solution. We use the SOS method. Write the inequality as follows:

\[
\sum \left[ 1 - \frac{4a^2 - b^2 - c^2}{a(b + c)} \right] \geq 0, \\
\sum \frac{b^2 + c^2 - 4a^2 + a(b + c)}{a(b + c)} \geq 0, \\
\sum \frac{(b^2 - a^2) + a(b - a) + (c^2 - a^2) + a(c - a)}{a(b + c)} \geq 0, \\
\sum \frac{(b - a)(2a + b) + (c - a)(2a + c)}{a(b + c)} \geq 0, \\
\sum \frac{(b - a)(2a + b) + (a - b)(2b + a)}{b(c + a)} \geq 0, \\
\sum c(a + b)(a - b)^2(bc + ca - ab) \geq 0.
\]

Without loss of generality, assume that \( a \geq b \geq c \). Since \( ca + ab - bc > 0 \), it suffices to show that

\[
b(c + a)(c - a)^2(ab + bc - ca) + c(a + b)(a - b)^2(bc + ca - ab) \geq 0,
\]

that is,

\[
b(c + a)(c - a)^2(ab + bc - ca) \geq c(a + b)(a - b)^2(ab - bc - ca).
\]

For the nontrivial case \( ab - bc - ca > 0 \), this inequality follows by multiplying the inequalities

\[
ab + bc - ca > ab - bc - ca, \\
(a - c)^2 \geq (a - b)^2, \\
b(c + a) \geq c(a + b).
\]

The equality holds for \( a = b = c \).

P 1.3. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

(a) \[
\frac{1}{a^2 + bc} + \frac{1}{b^2 + ca} + \frac{1}{c^2 + ab} \geq \frac{3}{ab + bc + ca};
\]

(b) \[
\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq \frac{2}{ab + bc + ca}.
\]

(c) \[
\frac{1}{a^2 + 2bc} + \frac{1}{b^2 + 2ca} + \frac{1}{c^2 + 2ab} > \frac{2}{ab + bc + ca}.
\]

(Vasile Cîrtoaje, 2005)
Symmetric Rational Inequalities

Solution. (a) Since
\[ \frac{ab + bc + ca}{a^2 + bc} = 1 + \frac{a(b + c - a)}{a^2 + bc}, \]
we can write the inequality as
\[ \frac{a(b + c - a)}{a^2 + bc} + \frac{b(c + a - b)}{b^2 + ca} + \frac{c(a + b - c)}{c^2 + ab} \geq 0. \]
Without loss of generality, assume that \( a = \min\{a, b, c\} \). Since \( b + c - a > 0 \), it suffices to show that
\[ \frac{b(c + a - b)}{b^2 + ca} + \frac{c(a + b - c)}{c^2 + ab} \geq 0. \]
This is equivalent to each of the following inequalities
\[ (b^2 + c^2)a^2 - (b + c)(b^2 - 3bc + c^2)a + bc(b - c)^2 \geq 0, \]
\[ (b - c)^2a^2 - (b + c)(b - c)^2a + bc(b - c)^2 + abc(2a + b + c) \geq 0, \]
\[ (b - c)^2(a - b)(a - c) + abc(2a + b + c) \geq 0. \]
The last inequality is obviously true. The equality holds for \( a = 0 \) and \( b = c \) (or any cyclic permutation thereof).

(b) According to the identities
\[ 2a^2 + bc = a(2a - b - c) + ab + bc + ca, \]
\[ 2b^2 + ca = b(2b - c - a) + ab + bc + ca, \]
\[ 2c^2 + ab = c(2c - a - b) + ab + bc + ca, \]
we can write the inequality as
\[ \frac{1}{1 + x} + \frac{1}{1 + y} + \frac{1}{1 + z} \geq 2, \]
where
\[ x = \frac{a(2a - b - c)}{ab + bc + ca}, \quad y = \frac{b(2b - c - a)}{ab + bc + ca}, \quad z = \frac{c(2c - a - b)}{ab + bc + ca}. \]
Without loss of generality, assume that \( a = \min\{a, b, c\} \). Since \( x \leq 0 \) and \( \frac{1}{1 + x} \geq 1 \), it suffices to show that
\[ \frac{1}{1 + y} + \frac{1}{1 + z} \geq 1. \]
This is equivalent to each of the following inequalities
\[ 1 \geq yz, \]
\[(ab + bc + ca)^2 \geq bc(2b - c - a)(2c - a - b),\]
\[a^2(b^2 + bc + c^2) + 3abc(b + c) + 2bc(b - c)^2 \geq 0.\]

The last inequality is obviously true. The equality holds for \(a = 0\) and \(b = c\) (or any cyclic permutation thereof).

(c) According to the identities
\[a^2 + 2bc = (a - b)(a - c) + ab + bc + ca,\]
\[b^2 + 2ca = (b - c)(b - a) + ab + bc + ca,\]
\[c^2 + 2ab = (c - a)(c - b) + ab + bc + ca,\]
we can write the inequality as
\[\frac{1}{1 + x} + \frac{1}{1 + y} + \frac{1}{1 + z} > 2,\]
where
\[x = \frac{(a - b)(a - c)}{ab + bc + ca}, \quad y = \frac{(b - c)(b - a)}{ab + bc + ca}, \quad z = \frac{(c - a)(c - b)}{ab + bc + ca}.\]

Since
\[xy + yz + zx = 0\]
and
\[xyz = \frac{-(a - b)^2(b - c)^2(c - a)^2}{(ab + bc + ca)^3} \leq 0,\]
we have
\[\frac{1}{1 + x} + \frac{1}{1 + y} + \frac{1}{1 + z} - 2 = \frac{1 - 2xyz}{(1 + x)(1 + y)(1 + z)} > 0.\]
\[\square\]

**P 1.4.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[\frac{a(b + c)}{a^2 + bc} + \frac{b(c + a)}{b^2 + ca} + \frac{c(a + b)}{c^2 + ab} \geq 2.\]

*(Pham Kim Hung, 2006)*

**Solution.** Without loss of generality, assume that \(a \geq b \geq c\) and write the inequality as
\[\frac{b(c + a)}{b^2 + ca} \geq \frac{(a - b)(a - c)}{a^2 + bc} + \frac{(a - c)(b - c)}{c^2 + ab}.\]
Symmetric Rational Inequalities

Since
\[
\frac{(a-b)(a-c)}{a^2 + bc} \leq \frac{(a-b)a}{a^2 + bc} \leq \frac{a-b}{a},
\]
and
\[
\frac{(a-c)(b-c)}{c^2 + ab} \leq \frac{a(b-c)}{c^2 + ab} \leq \frac{b-c}{b},
\]
it suffices to show that
\[
\frac{b(c+a)}{b^2 + ca} \geq \frac{a-b}{a} + \frac{b-c}{b}.
\]
This inequality is equivalent to
\[
b^2(a-b)^2 - 2abc(a-b) + a^2c^2 + ab^2c \geq 0,
\]
\[
(ab-b^2-ac)^2 + ab^2c \geq 0.
\]
The equality holds for for \(a = 0\) and \(b = c\) (or any cyclic permutation).

**P 1.5.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\sum \left( \frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \right) \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.
\]
(Vasile Cirtoaje, 2002)

**Solution.** We have
\[
\sum \left( \frac{a^2}{b^2 + c^2} - \frac{a}{b+c} \right) = \sum \frac{ab(a-b) + ac(a-c)}{(b^2 + c^2)(b+c)}
\]
\[
= \sum \frac{ab(a-b)}{(b^2 + c^2)(b+c)} + \sum \frac{ba(b-a)}{(c^2 + a^2)(c+a)}
\]
\[
= (a^2 + b^2 + c^2 + ab + bc + ca) \sum \frac{ab(a-b)^2}{(b^2 + c^2)(c^2 + a^2)(b+c)(c+a)} \geq 0.
\]
The equality holds for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation).

**P 1.6.** Let \(a, b, c\) be positive real numbers. Prove that
\[
\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{a}{a^2 + bc} + \frac{b}{b^2 + ca} + \frac{c}{c^2 + ab}.
\]
First Solution. Without loss of generality, assume that \( a = \min \{a, b, c\} \). Since
\[
\sum \frac{1}{b + c} - \sum \frac{a}{a^2 + bc} = \sum \left( \frac{1}{b + c} - \frac{a}{a^2 + bc} \right) = \sum \frac{(a - b)(a - c)}{(b + c)(a^2 + bc)}
\]
and \((a - b)(a - c) \geq 0\), it suffices to show that
\[
\frac{(b - c)(b - a)}{(c + a)(b^2 + ca)} + \frac{(c - a)(c - b)}{(a + b)(c^2 + ab)} \geq 0.
\]
This inequality is equivalent to
\[
(b - c)((b^2 - a^2)(c^2 + ab) + (a^2 - c^2)(b^2 + ca)) \geq 0,
\]
\[
a(b - c)^2(b^2 + c^2 - a^2 + ab + bc + ca) \geq 0,
\]
and is clearly true for \( a = \min \{a, b, c\} \). The equality holds for \( a = b = c \).

Second Solution. Since
\[
\sum \frac{1}{b + c} = \sum \left[ \frac{b}{(b + c)^2} + \frac{c}{(b + c)^2} \right] = \sum a \left[ \frac{1}{(a + b)^2} + \frac{1}{(a + c)^2} \right],
\]
we can write the inequality as
\[
\sum a \left[ \frac{1}{(a + b)^2} + \frac{1}{(a + c)^2} - \frac{1}{a^2 + bc} \right] \geq 0.
\]
This is true, since
\[
\frac{1}{(a + b)^2} + \frac{1}{(a + c)^2} - \frac{1}{a^2 + bc} = \frac{bc(b - c)^2 + (a^2 - bc)^2}{(a + b)^2(a + c)^2(a^2 + bc)} \geq 0.
\]
We can also prove this inequality using the Cauchy-Schwarz inequality, as follows
\[
\frac{1}{(a + b)^2} + \frac{1}{(a + c)^2} - \frac{1}{a^2 + bc} \geq \frac{(c + b)^2}{c^2(a + b)^2 + c^2(a + c)^2} \geq \frac{1}{a^2 + bc}
\]
\[
= \frac{bc[2a^2 - 2a(b + c) + b^2 + c^2]}{(a^2 + bc)[c^2(a + b)^2 + b^2(a + c)^2]} \geq \frac{bc[(2a - b - c)^2 + (b - c)^2]}{2(a^2 + bc)[c^2(a + b)^2 + b^2(a + c)^2]} \geq 0.
\]
P 1.7. Let \( a, b, c \) be positive real numbers. Prove that

\[
\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{2a}{3a^2 + bc} + \frac{2b}{3b^2 + ca} + \frac{2c}{3c^2 + ab}.
\]

(Vasile Cîrtoaje, 2005)

Solution. Since

\[
\sum \frac{1}{b+c} - \sum \frac{2a}{3a^2 + bc} = \sum \left( \frac{1}{b+c} - \frac{2a}{3a^2 + bc} \right) = \sum \frac{(a-b)(a-c) + a(2a-b-c)}{(b+c)(3a^2 + bc)},
\]

it suffices to show that

\[
\sum \frac{(a-b)(a-c)}{(b+c)(3a^2 + bc)} \geq 0
\]

and

\[
\sum a(2a-b-c) \geq 0.
\]

In order to prove the first inequality, assume that \( a = \min\{a, b, c\} \). Since

\[
(a-b)(a-c) \geq 0,
\]

it is enough to show that

\[
\frac{(b-c)(b-a)}{(c+a)(3b^2 + ca)} + \frac{(c-a)(c-b)}{(a+b)(3c^2 + ab)} \geq 0.
\]

This is equivalent to the obvious inequality

\[
a(b-c)^2(b^2 + c^2 - a^2 + 3ab + bc + 3ca) \geq 0.
\]

The second inequality is also true, since

\[
\sum \frac{a(2a-b-c)}{(b+c)(3a^2 + bc)} = \sum \frac{a(a-b) + a(a-c)}{(b+c)(3a^2 + bc)}
\]

\[
= \sum \frac{a(a-b)}{(b+c)(3a^2 + bc)} + \sum \frac{b(b-a)}{(a+b)(3b^2 + ca)}
\]

\[
= \sum (a-b) \left[ \frac{a}{(b+c)(3a^2 + bc)} - \frac{b}{(c+a)(3b^2 + ca)} \right]
\]

\[
= \sum \frac{c(a-b)^2[(a-b)^2 + c(a+b)]}{(b+c)(c+a)(3a^2 + bc)(3b^2 + ca)} \geq 0.
\]

The equality holds for \( a = b = c \). \qed
P 1.8. Let $a, b, c$ be positive real numbers. Prove that

(a) \[
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{13}{6} - \frac{2(ab + bc + ca)}{3(a^2 + b^2 + c^2)};
\]

(b) \[
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq (\sqrt{3} - 1)\left(1 - \frac{ab + bc + ca}{a^2 + b^2 + c^2}\right).
\]

Solution. (a) We use the SOS method. Rewrite the inequality as

\[
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq \frac{2}{3} \left(1 - \frac{ab + bc + ca}{a^2 + b^2 + c^2}\right).
\]

Since

\[
\sum\left(\frac{a}{b+c} - \frac{1}{2}\right) = \sum \left(\frac{a-b}{b+c}\right)
\]

\[
= \sum \left(\frac{a-b}{2(b+c)} + \frac{b-a}{2(c+a)}\right)
\]

\[
= \sum \left(\frac{a-b}{2}\left(\frac{1}{b+c} - \frac{1}{c+a}\right)\right)
\]

\[
= \sum \frac{(a-b)^2}{2(b+c)(c+a)}
\]

and

\[
\frac{2}{3} \left(1 - \frac{ab + bc + ca}{a^2 + b^2 + c^2}\right) = \sum \frac{(a-b)^2}{3(a^2 + b^2 + c^2)},
\]

the inequality can be restated as

\[
\sum(a-b)^2 \left[\frac{1}{2(b+c)(c+a)} - \frac{1}{3(a^2 + b^2 + c^2)}\right] \geq 0.
\]

This is true, since

\[
3(a^2 + b^2 + c^2) - 2(b+c)(c+a) = (a+b-c)^2 + 2(a-b)^2 \geq 0.
\]

The equality holds for $a = b = c$.

(b) Let

\[
p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.
\]

We have

\[
\sum \frac{a}{b+c} = \sum \left(\frac{1}{b+c} + 1\right) - 3 = p \sum \frac{1}{b+c} - 3
\]

\[
= \frac{p(p^2 + q)}{pq - r} - 3.
\]
According to P 2.57-(a) in Volume 1, for fixed \( p \) and \( q \), the product \( r \) is minimal when \( a = 0 \) or \( b = c \). Therefore, it suffices to prove the inequality for \( a = 0 \) and for \( b = c = 1 \).

**Case 1: \( a = 0 \).** The original inequality can be written as
\[
\frac{b}{c} + \frac{c}{b} - \frac{3}{2} \geq (\sqrt{3} - 1)
\left(1 - \frac{bc}{b^2 + c^2}\right).
\]
It suffices to show that
\[
\frac{b}{c} + \frac{c}{b} - \frac{3}{2} \geq 1 - \frac{bc}{b^2 + c^2}.
\]
Denoting
\[
t = \frac{b^2 + c^2}{bc}, \quad t \geq 2,
\]
this inequality becomes
\[
t - \frac{3}{2} \geq 1 - \frac{1}{t},
\]
\[(t - 2)(2t - 1) \geq 0.
\]

**Case 2: \( b = c = 1 \).** The original inequality becomes as follows:
\[
\frac{a}{2} + \frac{2}{a + 1} - \frac{3}{2} \geq (\sqrt{3} - 1)
\left(1 - \frac{2a + 1}{a^2 + 2}\right),
\]
\[
\frac{(a - 1)^2}{2(a + 1)} \geq \frac{(\sqrt{3} - 1)(a - 1)^2}{a^2 + 2},
\]
\[(a - 1)^2(1 - \sqrt{3} + 1)^2 \geq 0.
\]
The equality holds for \( a = b = c \), and for \( \frac{a}{\sqrt{3} - 1} = b = c \) (or any cyclic permutation).

\( \square \)

**P 1.9.** Let \( a, b, c \) be positive real numbers. Prove that
\[
\frac{1}{a^2 + 2bc} + \frac{1}{b^2 + 2ca} + \frac{1}{c^2 + 2ab} \leq \left(\frac{a + b + c}{ab + bc + ca}\right)^2.
\]
(\text{Vasile Cirtoaje, 2006})

**First Solution.** Write the inequality as
\[
\frac{(a + b + c)^2}{ab + bc + ca} - 3 \geq \sum \left(\frac{ab + bc + ca}{a^2 + 2bc} - 1\right),
\]
\[
\frac{(a-b)^2 + (b-c)^2 + (a-b)(b-c)}{ab + bc + ca} + \sum \frac{(a-b)(a-c)}{a^2 + 2bc} \geq 0.
\]

Without loss of generality, assume that \(a \geq b \geq c\). Since \((a-b)(a-c) \geq 0\) and \((c-a)(c-b) \geq 0\), it suffices to show that
\[
(a-b)^2 + (b-c)^2 + (a-b)(b-c) - \frac{(ab + bc + ca)(a-b)(b-c)}{b^2 + 2ca} \geq 0.
\]

This inequality is equivalent to
\[
(a-b)^2 + (b-c)^2 - \frac{(a-b)^2(b-c)^2}{b^2 + 2ca} \geq 0,
\]
or
\[
(b-c)^2 + \frac{c(a-b)^2(2a + 2b - c)}{b^2 + 2ca} \geq 0,
\]
which is clearly true. The equality holds for \(a = b = c\).

**Second Solution.** Assume that \(a \geq b \geq c\) and write the desired inequality as
\[
\frac{(a+b+c)^2}{ab + bc + ca} - 3 \geq \sum \left( \frac{ab + bc + ca}{a^2 + 2bc} - 1 \right),
\]
\[
\frac{1}{ab + bc + ca} \sum (a-b)(a-c) + \sum \frac{(a-b)(a-c)}{a^2 + 2bc} \geq 0,
\]
\[
\sum \left( 1 + \frac{ab + bc + ca}{a^2 + 2bc} \right) (a-b)(a-c) \geq 0.
\]

Since \((c-a)(c-b) \geq 0\) and \(a-b \geq 0\), it suffices to prove that
\[
\left( 1 + \frac{ab + bc + ca}{a^2 + 2bc} \right) (a-c) + \left( 1 + \frac{ab + bc + ca}{b^2 + 2ca} \right) (c-b) \geq 0.
\]

Write this inequality as
\[
a-b + (ab + bc + ca) \left( \frac{a-c}{a^2 + 2bc} + \frac{c-b}{b^2 + 2ca} \right) \geq 0,
\]
\[
(a-b) \left[ 1 + \frac{(ab + bc + ca)(3ac + 3bc - ab - 2c^2)}{(a^2 + 2bc)(b^2 + 2ca)} \right] \geq 0.
\]

Since \(a-b \geq 0\) and \(2ac + 3bc - 2c^2 > 0\), it is enough to show that
\[
1 + \frac{(ab + bc + ca)(ac-ab)}{(a^2 + 2bc)(b^2 + 2ca)} \geq 0.
\]

We have
\[
1 + \frac{(ab + bc + ca)(ac-ab)}{(a^2 + 2bc)(b^2 + 2ca)} \geq 1 + \frac{(ab + bc + ca)(ac-ab)}{a^2(b^2 + ca)}
\]
\[
= \frac{(a+b)c^2 + (a^2 - b^2)c}{a(b^2 + ca)} > 0.
\]

\(\blacksquare\)
P 1.10. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{a^2(b + c)}{b^2 + c^2} + \frac{b^2(c + a)}{c^2 + a^2} + \frac{c^2(a + b)}{a^2 + b^2} \geq a + b + c.
\]

(Darij Grinberg, 2004)

**First Solution.** We use the SOS method. We have

\[
\sum \frac{a^2(b + c)}{b^2 + c^2} - \sum a = \sum \left[ \frac{a^2(b + c)}{b^2 + c^2} - a \right]
\]

\[
= \sum \frac{ab(a - b) + ac(a - c)}{b^2 + c^2}
\]

\[
= \sum \frac{ab(a - b)}{b^2 + c^2} + \sum \frac{ba(b - a)}{c^2 + a^2}
\]

\[
= \sum \frac{ab(a + b)(a - b)^2}{(b^2 + c^2)(c^2 + a^2)} \geq 0.
\]

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

**Second Solution.** By virtue of the Cauchy-Schwarz inequality, we have

\[
\sum a^2(b + c) \geq \left( \sum a \right) \left( \sum a^2(b + c)(b^2 + c^2) \right).
\]

Then, it suffices to show that

\[
\left[ \sum a^2(b + c) \right]^2 \geq \left( \sum a \right) \left[ \sum a^2(b + c)(b^2 + c^2) \right].
\]

Let $p = a + b + c$ and $q = ab + bc + ca$. Since

\[
\left[ \sum a^2(b + c) \right]^2 = (pq - 3abc)^2
\]

\[
= p^2q^2 - 6abcqp + 9a^2b^2c^2
\]

and

\[
\sum a^2(b + c)(b^2 + c^2) = \sum (b + c)[(a^2b^2 + b^2c^2 + c^2a^2) - b^2c^2]
\]

\[
= 2p(a^2b^2 + b^2c^2 + c^2a^2) - \sum b^2c^2(p - a)
\]

\[
= p(a^2b^2 + b^2c^2 + c^2a^2) + abcq = p(q^2 - 2abc) + abcq,
\]

the inequality can be written as

\[
abc(2p^3 + 9abc - 7pq) \geq 0.
\]
Using Schur’s inequality
\[ p^3 + 9abc - 4pq \geq 0, \]
we have
\[ 2p^3 + 9abc - 7pq \geq p(p^2 - 3q) \geq 0. \]
\[
\square
\]

**P 1.11.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} \leq 3\left(\frac{a^2 + b^2 + c^2}{a + b + c}\right).
\]

**Solution.** We use the SOS method.

**First Solution.** Multiplying by \( 2(a + b + c) \), the inequality successively becomes
\[
\sum \left(1 + \frac{a}{b + c}\right)(b^2 + c^2) \leq 3(a^2 + b^2 + c^2),
\]
\[
\sum \frac{a}{b + c}(b^2 + c^2) \leq \sum a^2,
\]
\[
\sum a\left(a - \frac{b^2 + c^2}{b + c}\right) \geq 0,
\]
\[
\sum \frac{ab(a - b) - ac(c - a)}{b + c} \geq 0,
\]
\[
\sum \frac{ab(a - b) - ba(a - b)}{c + a} \geq 0,
\]
\[
\sum \frac{ab(a - b)^2}{(b + c)(c + a)} \geq 0.
\]
The equality holds for \( a = b = c \), and also for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

**Second Solution.** Subtracting \( a + b + c \) from the both sides, the desired inequality becomes as follows:
\[
\frac{3(a^2 + b^2 + c^2)}{a + b + c} - (a + b + c) \geq \sum \left(\frac{a^2 + b^2}{a + b} - \frac{a + b}{2}\right),
\]
\[
\sum \frac{(a - b)^2}{a + b + c} \geq \sum \frac{(a - b)^2}{2(a + b)},
\]
\[
\sum \frac{(a + b - c)(a - b)^2}{a + b} \geq 0.
\]
Without loss of generality, assume that \( a \geq b \geq c \). Since \( a + b - c \geq 0 \), it suffices to prove that
\[
\frac{(a + c - b)(a - c)^2}{a + c} \geq \frac{(a - b - c)(b - c)^2}{b + c}.
\]
This inequality is true because \( a + c - b \geq a - b - c \), \( a - c \geq b - c \) and \( \frac{a - c}{a + c} \geq \frac{b - c}{b + c} \).

**Third Solution.** Write the inequality as follows
\[
\sum \left[ \frac{3(a^2 + b^2)}{2(a + b + c)} - \frac{a^2 + b^2}{a + b} \right] \geq 0,
\]
\[
\sum \frac{(a^2 + b^2)(a + b - 2c)}{a + b} \geq 0,
\]
\[
\sum \frac{(a^2 + b^2)(a - c)}{a + b} + \sum \frac{(a^2 + b^2)(b - c)}{a + b} \geq 0,
\]
\[
\sum \frac{(a^2 + b^2)(a - c)}{a + b} + \sum \frac{(b^2 + c^2)(c - a)}{b + c} \geq 0,
\]
\[
\sum \frac{(a - c)^2(ab + bc + ca - b^2)}{(a + b)(b + c)} \geq 0.
\]
It suffices to prove that
\[
\sum \frac{(a - c)^2(ab + bc - ca - b^2)}{(a + b)(b + c)} \geq 0.
\]
Since \( ab + bc - ca - b^2 = (a - b)(b - c) \), this inequality is equivalent to
\[
(a - b)(b - c)(c - a) \sum \frac{c - a}{(a + b)(b + c)} \geq 0,
\]
which is true because
\[
\sum \frac{c - a}{(a + b)(b + c)} = 0.
\]

P 1.12. Let \( a, b, c \) be positive real numbers. Prove that
\[
\frac{1}{a^2 + ab + b^2} + \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \geq \frac{9}{(a + b + c)^2}.
\]

*(Vasile Cîrtoaje, 2000)*
First Solution. Due to homogeneity, we may assume that \( a + b + c = 1 \). Let \( q = ab + bc + ca \). Since
\[
b^2 + bc + c^2 = (a + b + c)^2 - a(a + b + c) - (ab + bc + ca) = 1 - a - q,
\]
we can write the inequality as
\[
\sum \frac{1}{1 - a - q} \geq 9,
\]
\[
9q^3 - 6q^2 - 3q + 1 + 9abc \geq 0.
\]
From Schur’s inequality
\[
(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),
\]
we get \( 1 + 9abc - 4q \geq 0 \). Therefore,
\[
9q^3 - 6q^2 - 3q + 1 + 9abc = (1 + 9abc - 4q) + q(3q - 1)^2 \geq 0.
\]
The equality holds for \( a = b = c \).

Second Solution. Multiplying by \( a^2 + b^2 + c^2 + ab + bc + ca \), the inequality can be written as
\[
(a + b + c) \sum \frac{a}{b^2 + bc + c^2} + \frac{9(ab + bc + ca)}{(a + b + c)^2} \geq 6.
\]
By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{a}{b^2 + bc + c^2} \geq \frac{(a + b + c)^2}{\sum a(b^2 + bc + c^2)} = \frac{a + b + c}{ab + bc + ca}.
\]
Then, it suffices to show that
\[
\frac{(a + b + c)^2}{ab + bc + ca} + \frac{9(ab + bc + ca)}{(a + b + c)^2} \geq 6.
\]
This follows immediately from the AM-GM inequality.

\[\square\]

P 1.13. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2}{(2a + b)(2a + c)} + \frac{b^2}{(2b + c)(2b + a)} + \frac{c^2}{(2c + a)(2c + b)} \leq \frac{1}{3}.
\]

(Tigran Sloyan, 2005)
**First Solution.** The inequality is equivalent to each of the inequalities

\[
\sum \left[ \frac{a^2}{(2a+b)(2a+c)} - \frac{a}{3(2a+b+c)} \right] \leq 0,
\]

\[
\sum \frac{a(a-b)(a-c)}{(2a+b)(2a+c)} \geq 0.
\]

Due to symmetry, we may consider that \(a \geq b \geq c\). Since \(c(c-a)(c-b) \geq 0\), it suffices to prove that

\[
\frac{a(a-b)(a-c)}{(2a+b)(2a+c)} + \frac{b(b-c)(b-a)}{(2b+c)(2b+a)} \geq 0.
\]

This is equivalent to the obvious inequality

\[
(a-b)^2[(a+b)(2ab-c^2) + c(a^2 + b^2 + 5ab)] \geq 0.
\]

The equality holds for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation).

**Second Solution** (by Vo Quoc Ba Can). Apply the Cauchy-Schwarz inequality in the following manner

\[
\frac{9a^2}{(2a+b)(2a+c)} = \frac{(2a+a)^2}{2a(a+b+c) + (2a^2 + bc)} \leq \frac{2a}{a+b+c} + \frac{a^2}{2a^2 + bc}.
\]

Then,

\[
\sum \frac{9a^2}{(2a+b)(2a+c)} \leq 2 + \sum \frac{a^2}{2a^2 + bc},
\]

and from the known inequality

\[
\sum \frac{a^2}{2a^2 + bc} \leq 1,
\]

the conclusion follows. The last inequality is equivalent to

\[
\sum \frac{bc}{2a^2 + bc} \geq 1,
\]

and can be obtained using the Cauchy-Schwarz inequality, as follows

\[
\sum \frac{bc}{2a^2 + bc} \geq \frac{(\sum bc)^2}{\sum bc(2a^2 + bc)} = 1.
\]

**Remark.** From the inequality in P 1.13 and Hölder's inequality

\[
\left[ \sum \frac{a^2}{(2a+b)(2a+c)} \right] \left[ \sum \sqrt{a(2a+b)(2a+c)} \right]^2 \geq (a+b+c)^3,
\]
we get the following result:

- If $a, b, c$ are nonnegative real numbers such that $a + b + c = 3$, then
  \[
  \sqrt{a(2a + b)(2a + c)} + \sqrt{b(2b + c)(2b + a)} + \sqrt{c(2c + a)(2c + bc)} \geq 9,
  \]
  with equality for $a = b = c = 1$, and for $(a, b, c) = (0, \frac{3}{2}, \frac{3}{2})$ or any cyclic permutation.

\[ \square \]

P 1.14. Let $a, b, c$ be positive real numbers. Prove that

\[(a) \quad \sum \frac{a}{(2a + b)(2a + c)} \leq \frac{1}{a + b + c};
(b) \quad \sum \frac{a^3}{(2a^2 + b^2)(2a^2 + c^2)} \leq \frac{1}{a + b + c}.
\]

(Vasile Cîrtoaje, 2005)

**Solution.** (a) Write the inequality as

\[
\sum \left[ \frac{1}{3} - \frac{a(a + b + c)}{(2a + b)(2a + c)} \right] \geq 0,
\]

\[
\sum \frac{(a - b)(a - c)}{(2a + b)(2a + c)} \geq 0.
\]

Assume that $a \geq b \geq c$. Since $(a - b)(a - c) \geq 0$, it suffices to prove that

\[
\frac{(b - c)(b - a)}{(2b + c)(2b + a)} + \frac{(a - c)(b - c)}{(2c + a)(2c + b)} \geq 0.
\]

Since $b - c \geq 0$ and $a - c \geq a - b \geq 0$, it is enough to show that

\[
\frac{1}{(2c + a)(2c + b)} \geq \frac{1}{(2b + c)(2b + a)}.
\]

This is equivalent to the obvious inequality

\[(b - c)(a + 4b + 4c) \geq 0.
\]

The equality holds for $a = b = c$.

(b) We obtain the desired inequality by summing the inequalities

\[
\frac{a^3}{(2a^2 + b^2)(2a^2 + c^2)} \geq \frac{a}{(a + b + c)^2}.
\]
Symmetric Rational Inequalities

\[
\frac{b^3}{(2b^2 + c^2)(2b^2 + a^2)} \geq \frac{b}{(a + b + c)^2},
\]

\[
\frac{c^3}{(2c^2 + a^2)(2c^2 + b^2)} \geq \frac{c}{(a + b + c)^2},
\]

which are consequences of the Cauchy-Schwarz inequality. For example, from\[
(a^2 + a^2 + b^2)(c^2 + a^2 + a^2) \geq (ac + a^2 + ba)^2,
\]
the first inequality follows. The equality holds for \(a = b = c\).

\[\Box\]

**P 1.15.** If \(a, b, c\) are positive real numbers, then

\[
\sum \frac{1}{(a + 2b)(a + 2c)} \geq \frac{1}{(a + b + c)^2} + \frac{2}{3(ab + bc + ca)}.
\]

**Solution.** Write the inequality as follows

\[
\sum \left[ \frac{1}{(a + 2b)(a + 2c)} - \frac{1}{(a + b + c)^2} \right] \geq \frac{2}{3(ab + bc + ca)} - \frac{2}{(a + b + c)^2},
\]

\[
\sum \frac{(b - c)^2}{(a + 2b)(a + 2c)} \geq \sum \frac{(b - c)^2}{3(ab + bc + ca)},
\]

\[(a - b)(b - c)(c - a) \sum \frac{b - c}{(a + 2b)(a + 2c)} \geq 0.
\]

Since

\[
\sum \frac{b - c}{(a + 2b)(a + 2c)} = \sum \left[ \frac{b - c}{(a + 2b)(a + 2c)} - \frac{b - c}{3(ab + bc + ca)} \right] = \frac{(a - b)(b - c)(c - a)}{3(ab + bc + ca)} \sum \frac{1}{(a + 2b)(a + 2c)},
\]

the desired inequality is equivalent to the obvious inequality

\[(a - b)^2(b - c)^2(c - a)^2 \sum \frac{1}{(a + 2b)(a + 2c)} \geq 0.
\]

The equality holds for \(a = b\), or \(b = c\), or \(c = a\).

\[\Box\]
P 1.16. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

(a) \[
\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} \geq \frac{4}{ab + bc + ca};
\]

(b) \[
\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \geq \frac{3}{ab + bc + ca};
\]

(c) \[
\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{5}{2(ab + bc + ca)}.
\]

Solution. Let

\[
E_k(a, b, c) = \frac{ab + bc + ca}{a^2 - kab + b^2} + \frac{ab + bc + ca}{b^2 - kbc + c^2} + \frac{ab + bc + ca}{c^2 - kca + a^2}.
\]

We will prove that

\[
E_k(a, b, c) \geq \alpha_k,
\]

where

\[
\alpha_k = \begin{cases} 
5 - 2k, & 0 \leq k \leq 1 \\
2 - k, & 1 \leq k \leq 2 \\
\end{cases}
\]

To show this, assume that \(a \leq b \leq c\) and prove that

\[
E_k(a, b, c) \geq E_k(0, b, c) \geq \alpha_k.
\]

For the nontrivial case \(a > 0\), the left inequality is true because

\[
\frac{E_k(a, b, c) - E_k(0, b, c)}{a} = \frac{b^2 + (1+k)bc - ac}{b(a^2 - kab + b^2)} + \frac{b + c}{b^2 - kbc + c^2} + \frac{c^2 + (1+k)bc - ab}{c(c^2 - kca + a^2)}
\]

\[
> \frac{bc - ac}{b(a^2 - kab + b^2)} + \frac{b + c}{b^2 - kbc + c^2} + \frac{bc - ab}{c(c^2 - kca + a^2)} > 0.
\]

In order to prove the right inequality \(E_k(0, b, c) \geq \alpha_k\), where

\[
E_k(0, b, c) = \frac{bc}{b^2 - kbc + c^2} + \frac{b + c}{bc} + \frac{c}{b},
\]

by virtue of the AM-GM inequality, we have

\[
E_k(0, b, c) = \frac{bc}{b^2 - kbc + c^2} + \frac{b^2 - kbc + c^2}{bc} + k \geq 2 + k.
\]
Thus, it remains to consider the case $0 \leq k \leq 1$. We have

$$E_k(0, b, c) = \frac{bc}{b^2 - kbc + c^2} + \frac{b^2 - kbc + c^2}{(2 - k)^2 bc} + \left[1 - \frac{1}{(2 - k)^2}\right] \left(\frac{c}{b} + \frac{b}{c}\right) + \frac{k}{(2 - k)^2} \geq \frac{2}{2 - k} + 2 \left[1 - \frac{1}{(2 - k)^2}\right] + \frac{k}{(2 - k)^2} = \frac{5 - 2k}{2 - k}.$$  

For $1 \leq k \leq 2$, the equality holds when $a = 0$ and $\frac{b}{c} + \frac{c}{b} = 1 + k$ (or any cyclic permutation). For $0 \leq k \leq 1$, the equality holds when $a = 0$ and $b = c$ (or any cyclic permutation).

\[\square\]

**P 1.17.** Let $a, b, c$ be positive real numbers, no two of which are zero. Prove that

$$\frac{(a^2 + b^2)(a^2 + c^2)}{(a + b)(a + c)} + \frac{(b^2 + c^2)(b^2 + a^2)}{(b + c)(b + a)} + \frac{(c^2 + a^2)(c^2 + b^2)}{(c + a)(c + b)} \geq a^2 + b^2 + c^2.$$  

(Vasile Cîrtoaje, 2011)

**Solution.** Using the identity

$$(a^2 + b^2)(a^2 + c^2) = a^2(a^2 + b^2 + c^2) + b^2c^2,$$

we can write the inequality as follows

$$\sum \frac{b^2c^2}{(a + b)(a + c)} \geq (a^2 + b^2 + c^2) \left[1 - \sum \frac{a^2}{(a + b)(a + c)}\right],$$

$$\sum b^2c^2(b + c) \geq 2abc(a^2 + b^2 + c^2),$$

$$\sum a^3(b^2 + c^2) \geq 2 \sum a^3bc,$$

$$\sum a^3(b - c)^2 \geq 0.$$  

Since the last form is obvious, the proof is completed. The equality holds for $a = b = c$.

\[\square\]

**P 1.18.** Let $a, b, c$ be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^2 + b + c} + \frac{1}{b^2 + c + a} + \frac{1}{c^2 + a + b} \leq 1.$$
First Solution. By virtue of the Cauchy-Schwarz inequality, we have

\[(a^2 + b + c)(1 + b + c) \geq (a + b + c)^2.\]

Therefore,

\[\sum \frac{1}{a^2 + b + c} \leq \sum \frac{1 + b + c}{(a + b + c)^2} = \frac{3 + 2(a + b + c)}{(a + b + c)^2} = 1.\]

The equality occurs for \(a = b = c = 1.\)

Second Solution. Rewrite the inequality as

\[\frac{1}{a^2 - a + 3} + \frac{1}{b^2 - b + 3} + \frac{1}{c^2 - c + 3} \leq 1.\]

We see that the equality holds for \(a = b = c = 1.\) Thus, if there exists a real number \(k\) such that

\[\frac{1}{a^2 - a + 3} \leq k + \left(\frac{1}{3} - k\right)a\]

for all \(a \in [0,3],\) then

\[\sum \frac{1}{a^2 - a + 3} \leq \sum \left[k + \left(\frac{1}{3} - k\right)a\right] = 3k + \left(\frac{1}{3} - k\right)\sum a = 1.\]

We have

\[k + \left(\frac{1}{3} - k\right)a - \frac{1}{a^2 - a + 3} = \frac{(a - 1)[(1 - 3k)a^2 + 3ka + 3(1 - 3k)]}{3(a^2 - a + 3)}.\]

Setting \(k = 4/9,\) we get

\[k + \left(\frac{1}{3} - k\right)a - \frac{1}{a^2 - a + 3} = \frac{(a - 1)^2(3 - a)}{9(a^2 - a + 3)} \geq 0.\]

\(\Box\)

P 1.19. Let \(a, b, c\) be real numbers such that \(a + b + c = 3.\) Prove that

\[\frac{a^2 - bc}{a^2 + 3} + \frac{b^2 - ca}{b^2 + 3} + \frac{c^2 - ab}{c^2 + 3} \geq 0.\]

(Vasile Cirtoaje, 2005)
Solution. Apply the SOS method. We have

\[
2 \sum \frac{a^2 - bc}{a^2 + 3} = \sum \frac{(a - b)(a + c) + (a - c)(a + b)}{a^2 + 3} = \sum \frac{(a - b)(a + c)}{a^2 + 3} + \sum \frac{(b - a)(b + c)}{b^2 + 3} = \sum (a - b) \left( \frac{a + c}{a^2 + 3} - \frac{b + c}{b^2 + 3} \right) = (3 - ab - bc - ca) \sum \frac{(a - b)^2}{(a^2 + 3)(b^2 + 3)} \geq 0.
\]

Thus, it suffices to show that \(3 - ab - bc - ca \geq 0\). This follows immediately from the known inequality \((a + b + c)^2 \geq 3(ab + bc + ca)\), which is equivalent to \((a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0\). The equality holds for \(a = b = c = 1\).

P 1.20. Let \(a, b, c\) be nonnegative real numbers such that \(a + b + c = 3\). Prove that

\[
\frac{1 - bc}{5 + 2a} + \frac{1 - ca}{5 + 2b} + \frac{1 - ab}{5 + 2c} \geq 0.
\]

Solution. Since

\[
9(1 - bc) = (a + b + c)^2 - 9bc,
\]

we can write the inequality as

\[
\sum \frac{a^2 + b^2 + c^2 + 2a(b + c) - 7bc}{5 + 2a} \geq 0.
\]

From

\[
(a - b)(a + kb + mc) + (a - c)(a + kc + mb) = 2a^2 - k(b^2 + c^2) + (k + m - 1)a(b + c) - 2mbc,
\]

choosing \(k = -2\) and \(m = 7\), we get

\[
(a - b)(a - 2b + 7c) + (a - c)(a - 2c + 7b) = 2[a^2 + b^2 + c^2 + 2a(b + c) - 7bc].
\]

Therefore, the desired inequality becomes as follows:

\[
\sum \frac{(a - b)(a - 2b + 7c)}{5 + 2a} + \sum \frac{(a - c)(a - 2c + 7b)}{5 + 2a} \geq 0,
\]

\[
\sum \frac{(a - b)(a - 2b + 7c)}{5 + 2a} + \sum \frac{(b - a)(b - 2a + 7c)}{5 + 2b} \geq 0,
\]
\[
\begin{align*}
\sum (a-b)(5+2c)[(5+2b)(a-2b+7c)-(5+2a)(b-2a+7c)] & \geq 0, \\
\sum (a-b)^2(5+2c)(15+4a+4b-14c) & \geq 0, \\
\sum (a-b)^2(5+2c)(a+b-c) & \geq 0.
\end{align*}
\]

Without loss of generality, assume that \(a \geq b \geq c\). Clearly, it suffices to show that
\[
(a-c)^2(5+2b)(a+c-b) \geq (b-c)^2(5+2a)(a-b-c).
\]
Since \(a-c \geq b-c \geq 0\) and \(a+c-b \geq a-b-c\), it suffices to prove that
\[
(a-c)(5+2b) \geq (b-c)(5+2a).
\]
Indeed,
\[
(a-c)(5+2b)-(b-c)(5+2a) = (a-b)(5+2c) \geq 0.
\]
The equality holds for \(a = b = c = 1\), and for \(c = 0\) and \(a = b = 3/2\) (or any cyclic permutation).

P 1.21. Let \(a, b, c\) be positive real numbers such that \(a + b + c = 3\). Prove that
\[
\frac{1}{a^2 + b^2 + 2} + \frac{1}{b^2 + c^2 + 2} + \frac{1}{c^2 + a^2 + 2} \leq \frac{3}{4}.
\]

\(\text{(Vasile Cirtoaje, 2006)}\)

Solution. Since
\[
\frac{1}{a^2 + b^2 + 2} = \frac{1}{2} - \frac{a^2 + b^2}{a^2 + b^2 + 2},
\]
we write the inequality as
\[
\frac{a^2 + b^2}{a^2 + b^2 + 2} + \frac{b^2 + c^2}{b^2 + c^2 + 2} + \frac{c^2 + a^2}{c^2 + a^2 + 2} \geq \frac{3}{2}.
\]
By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{a^2 + b^2}{a^2 + b^2 + 2} \geq \frac{(\sum \sqrt{a^2 + b^2})^2}{\sum (a^2 + b^2 + 2)} = \frac{2\sum a^2 + 2\sum \sqrt{(a^2 + b^2)(a^2 + c^2)}}{2\sum a^2 + 6} \geq \frac{2\sum a^2 + 2\sum (a^2 + bc)}{2\sum a^2 + 6} = \frac{3\sum a^2 + 9}{2\sum a^2 + 6} = \frac{3}{2}.
\]
The equality holds for \(a = b = c = 1\)
P 1.22. Let $a, b, c$ be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{4a^2 + b^2 + c^2} + \frac{1}{4b^2 + c^2 + a^2} + \frac{1}{4c^2 + a^2 + b^2} \leq \frac{1}{2}. \quad \text{(Vasile Cirtoaje, 2007)}$$

**Solution.** According to the Cauchy-Schwarz inequality, we have

$$\frac{9}{4a^2 + b^2 + c^2} = \frac{(a + b + c)^2}{2a^2 + (a^2 + b^2) + (a^2 + c^2)} \leq \frac{1}{2 + \frac{b^2}{a^2 + b^2} + \frac{c^2}{a^2 + c^2}}.$$

Therefore,

$$\sum \frac{9}{4a^2 + b^2 + c^2} \leq \frac{3}{2} + \sum \left( \frac{b^2}{a^2 + b^2} + \frac{c^2}{a^2 + c^2} \right) = \frac{3}{2} + \sum \left( \frac{b^2}{a^2 + b^2} + \frac{a^2}{b^2 + a^2} \right) = \frac{3}{2} + 3 = \frac{9}{2}.$$

The equality holds for $a = b = c = 1$. 

\[\square\]

P 1.23. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 2$. Prove that

$$\frac{bc}{a^2 + 1} + \frac{ca}{b^2 + 1} + \frac{ab}{c^2 + 1} \leq 1. \quad \text{(Pham Kim Hung, 2005)}$$

**Solution.** Let $p = a + b + c$ and $q = ab + bc + ca$, $q \leq p^2/3 = 4/3$. If one of $a, b, c$ is zero, then the inequality is true. Otherwise, write the inequality as

$$\sum \frac{1}{a(a^2 + 1)} \leq \frac{1}{abc},$$

$$\sum \left( \frac{1}{a} - \frac{a}{a^2 + 1} \right) \leq \frac{1}{abc},$$

$$\sum \frac{a}{a^2 + 1} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{abc},$$

$$\sum \frac{a}{a^2 + 1} \geq \frac{q - 1}{abc},$$
Using the inequality
\[ \frac{2}{a^2 + 1} \geq 2 - a, \]
which is equivalent to \( a(a - 1)^2 \geq 0 \), we get
\[ \sum \frac{a}{a^2 + 1} \geq \sum \frac{a(2 - a)}{2} = \sum \frac{a(b + c)}{2} = q. \]

Therefore, it suffices to prove that
\[ 1 + abcq \geq q. \]

Since
\[
\begin{align*}
    a^4 + b^4 + c^4 &= (a^2 + b^2 + c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) \\
    &= (p^2 - 2q)^2 - 2(q^2 - 2abcp) = p^4 - 4p^2q + 2q^2 + 4abcp
\end{align*}
\]
by Schur’s inequality of degree four
\[ a^4 + b^4 + c^4 + 2abc(a + b + c) \geq (ab + bc + ca)(a^2 + b^2 + c^2), \]
we get
\[ abc \geq \frac{(p^2 - q)(4q - p^2)}{6p}, \]
\[ abc \geq \frac{(4 - q)(q - 1)}{3}. \]

Thus
\[ 1 + abcq - q \geq 1 + \frac{q(4 - q)(q - 1)}{3} - q = \frac{(3 - q)(q - 1)^2}{3} \geq 0. \]
The equality holds if \( a = 0 \) and \( b = c = 1 \) (or any cyclic permutation).

\[ \square \]

**P 1.24.** Let \( a, b, c \) be nonnegative real numbers such that \( a + b + c = 1 \). Prove that
\[ \frac{bc}{a + 1} + \frac{ca}{b + 1} + \frac{ab}{c + 1} \leq \frac{1}{4}. \]

\[(Vasile Cirtoaje, 2009)\]
Symmetric Rational Inequalities

First Solution. We have
\[
\sum_{cyc} \frac{bc}{a+1} = \sum_{cyc} \frac{bc}{(a+b)+(c+a)} \\
\leq \frac{1}{4} \sum_{cyc} bc \left( \frac{1}{a+b} + \frac{1}{c+a} \right) \\
= \frac{1}{4} \sum_{cyc} bc + \frac{1}{4} \sum_{cyc} bc \\
= \frac{1}{4} \sum_{cyc} \frac{bc}{a+b} + \frac{1}{4} \sum_{cyc} \frac{ca}{a+b} \\
= \frac{1}{4} \sum_{cyc} \frac{bc+ca}{a+b} = \frac{1}{4} \sum_{cyc} c = \frac{1}{4}.
\]
The equality holds for \(a = b = c = 1/3\), and for \(a = 0\) and \(b = c = 1/2\) (or any cyclic permutation).

Second Solution. It is easy to check that the inequality is true if one of \(a, b, c\) is zero. Otherwise, write the inequality as
\[
\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \leq \frac{1}{4abc}.
\]
Since
\[
\frac{1}{a(a+1)} = \frac{1}{a} - \frac{1}{a+1},
\]
we can write the required inequality as
\[
\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{4abc}.
\]
In virtue of the Cauchy-Schwarz inequality,
\[
\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{9}{(a+1)+(b+1)+(c+1)} = \frac{9}{4}.
\]
Therefore, it suffices to prove that
\[
\frac{9}{4} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{4abc}.
\]
This is equivalent to Schur’s inequality
\[
(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca).
\]
P 1.25. Let \( a, b, c \) be positive real numbers such that \( a + b + c = 1 \). Prove that
\[
\frac{1}{a(2a^2 + 1)} + \frac{1}{b(2b^2 + 1)} + \frac{1}{c(2c^2 + 1)} \leq \frac{3}{11abc}.
\]
(Vasile Cîrtoaje, 2009)

Solution. Since
\[
\frac{1}{a(2a^2 + 1)} = \frac{1}{a} - \frac{2a}{2a^2 + 1},
\]
we can write the inequality as
\[
\sum \frac{2a}{2a^2 + 1} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{11abc}.
\]
By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{2a}{2a^2 + 1} \geq \frac{2(\sum a)^2}{\sum a(2a^2 + 1)} = \frac{2}{2(a^3 + b^3 + c^3) + 1}.
\]
Therefore, it suffices to show that
\[
\frac{2}{2(a^3 + b^3 + c^3) + 1} \geq \frac{11q - 3}{11abc},
\]
where \( q = ab + bc + ca, q \leq \frac{1}{3}(a + b + c)^2 = \frac{1}{3} \). Since
\[
a^3 + b^3 + c^3 = 3abc + (a + b + c)^3 - 3(a + b + c)(ab + bc + ca) = 3abc + 1 - 3q,
\]
we need to prove that
\[
22abc \geq (11q - 3)(6abc + 3 - 6q),
\]
or, equivalently,
\[
2(20 - 33q)abc \geq 3(11q - 3)(1 - 2q).
\]
From Schur’s inequality
\[
(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),
\]
we get
\[
9abc \geq 4q - 1.
\]
Thus,
\[
2(20 - 33q)abc - 3(11q - 3)(1 - 2q) \geq
\]
\[
\geq \frac{2(20 - 33q)(4q - 1)}{9} - 3(11q - 3)(1 - 2q)
\]
\[
= \frac{330q^2 - 233q + 41}{9} = \frac{(1 - 3q)(41 - 110q)}{9} \geq 0.
\]
This completes the proof. The equality holds for \( a = b = c = 1/3 \). \( \Box \)
P 1.26. Let \( a, b, c \) be positive real numbers such that \( a + b + c = 3 \). Prove that

\[
\frac{1}{a^3 + b + c} + \frac{1}{b^3 + c + a} + \frac{1}{c^3 + a + b} \leq 1.
\]

\textit{(Vasile Cirtoaje, 2009)}

\textbf{Solution.} Write the inequality in the form

\[
\frac{1}{a^3 - a + 3} + \frac{1}{b^3 - b + 3} + \frac{1}{c^3 - c + 3} \leq 1.
\]

Assume that \( a \geq b \geq c \). There are two cases to consider.

\textit{Case 1:} \( c \leq b \leq a \leq 2 \). The desired inequality follows by adding the inequalities

\[
\frac{1}{a^3 - a + 3} \leq \frac{5 - 2a}{9}, \quad \frac{1}{b^3 - b + 3} \leq \frac{5 - 2b}{9}, \quad \frac{1}{c^3 - c + 3} \leq \frac{5 - 2c}{9}.
\]

Indeed, we have

\[
\frac{1}{a^3 - a + 3} - \frac{5 - 2a}{9} = \frac{(a - 1)^2(a - 2)(2a + 3)}{9(a^3 - a + 3)} \leq 0.
\]

\textit{Case 2:} \( a > 2 \). From \( a + b + c = 3 \), we get \( b + c < 1 \). Since

\[
\sum a^3 - a + 3 < \frac{1}{a^3 - a + 3} + \frac{1}{3 - b} + \frac{1}{3 - c} < \frac{1}{9} + \frac{1}{3 - b} + \frac{1}{3 - c},
\]

it suffices to prove that

\[
\frac{1}{3 - b} + \frac{1}{3 - c} \leq \frac{8}{9}.
\]

We have

\[
\frac{1}{3 - b} + \frac{1}{3 - c} - \frac{8}{9} = \frac{3 - (1 - b - c) - 8bc}{9(3 - b)(3 - c)} < 0.
\]

The equality holds for \( a = b = c = 1 \).

\textbf{P 1.27.} Let \( a, b, c \) be positive real numbers such that \( a + b + c = 3 \). Prove that

\[
\frac{a^2}{1 + b^3 + c^3} + \frac{b^2}{1 + c^3 + a^3} + \frac{c^2}{1 + a^3 + b^3} \geq 1.
\]
**Solution.** Using the Cauchy-Schwarz inequality, we have

\[
\sum \frac{a^2}{1 + b^3 + c^3} \geq \frac{(\sum a^2)^2}{\sum a^2 (1 + b^3 + c^3)},
\]

and it remains to show that

\[
(a^2 + b^2 + c^2)^2 \geq (a^2 + b^2 + c^2) + \sum a^2 b^2 (a + b).
\]

Let \( p = a + b + c \) and \( q = ab + bc + ca, q \leq 3 \). Since \( a^2 + b^2 + c^2 = 9 - 2q \) and

\[
\sum a^2 b^2 (a + b) = 3 \sum a^2 b^2 - q abc = 3q^2 -(q + 18)abc,
\]

the desired inequality can be written as

\[
q^2 - 34q + 72 + (q + 18)abc \geq 0.
\]

This inequality is clearly true for \( q \leq 2 \). Consider further that \( 2 < q \leq 3 \). Since

\[
a^4 + b^4 + c^4 = (a^2 + b^2 + c^2)^2 - 2(a^2 b^2 + b^2 c^2 + c^2 a^2)
\]

\[
= (p^2 - 2q^2)^2 - 2(q^2 - 2abc) = p^4 - 4p^2 q + 4q^2 + 4abc
\]

by Schur's inequality of degree four

\[
a^4 + b^4 + c^4 + 2abc(a + b + c) \geq (ab + bc + ca)(a^2 + b^2 + c^2),
\]

we get

\[
abc \geq \frac{(p^2 - q)(4q - p^2)}{6p} = \frac{(9 - q)(4q - 9)}{18}.
\]

Therefore

\[
q^2 - 34q + 72 + (q + 18)abc \geq q^2 - 34q + 72 + \frac{(q + 18)(9 - q)(4q - 9)}{18}
\]

\[
= \frac{(3 - q)(4q^2 + 21q - 54)}{18} \geq 0.
\]

The equality holds for \( a = b = c = 1 \). 

\[ \square \]

**P 1.28.** Let \( a, b, c \) be nonnegative real numbers such that \( a + b + c = 3 \). Prove that

\[
\frac{1}{6 - ab} + \frac{1}{6 - bc} + \frac{1}{6 - ca} \leq \frac{3}{5}.
\]
**Solution.** Rewrite the inequality as

\[
108 - 48(ab + bc + ca) + 13abc(a + b + c) - 3a^2 b^2 c^2 \geq 0,
\]

\[
4[9 - 4(ab + bc + ca) + 3abc] + abc(1 - abc) \geq 0.
\]

By the AM-GM inequality,

\[
1 = \left(\frac{a + b + c}{3}\right)^3 \geq abc.
\]

Consequently, it suffices to show that

\[
9 - 4(ab + bc + ca) + 3abc \geq 0.
\]

We see that the homogeneous form of this inequality is just Schur’s inequality of third degree

\[
(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca).
\]

The equality holds for \(a = b = c = 1\), as well as for \(a = 0\) and \(b = c = 3/2\) (or any cyclic permutation).

\[
\square
\]

**P 1.29.** Let \(a, b, c\) be nonnegative real numbers such that \(a + b + c = 3\). Prove that

\[
\frac{1}{2a^2 + 7} + \frac{1}{2b^2 + 7} + \frac{1}{2c^2 + 7} \leq \frac{1}{3}.
\]

*(Vasile Cîrtoaje, 2005)*

**Solution.** Assume that \(a = \max\{a, b, c\}\) and prove that

\[
E(a, b, c) \leq E(a, s, s) \leq \frac{1}{3},
\]

where

\[
s = \frac{b + c}{2}, \quad 0 \leq s \leq 1,
\]

and

\[
E(a, b, c) = \frac{1}{2a^2 + 7} + \frac{1}{2b^2 + 7} + \frac{1}{2c^2 + 7}.
\]

We have

\[
E(a, s, s) - E(a, b, c) = \left(\frac{1}{2s^2 + 7} - \frac{1}{2b^2 + 7}\right) + \left(\frac{1}{2s^2 + 7} - \frac{1}{2c^2 + 7}\right)
\]

\[
= \frac{1}{2s^2 + 7} \left(\frac{(b - c)(b + s)}{2b^2 + 7} + \frac{(c - b)(c + s)}{2c^2 + 7}\right)
\]

\[
= \frac{(b - c)^2 (7 - 4s^2 - 2bc)}{(2s^2 + 7)(2b^2 + 7)(2c^2 + 7)}.
\]
Since $bc \leq s^2 \leq 1$, it follows that $7 - 4s^2 - 2bc > 0$, and hence $E(a, s, s) \geq E(a, b, c)$. Also,

$$
\frac{1}{3} - E(a, s, s) = \frac{1}{3} - E(3 - 2s, s, s) = \frac{4(s-1)^2(2s-1)^2}{3(2a^2 + 7)(2s^2 + 7)} \geq 0.
$$

The equality holds for $a = b = c = 1$, as well as for $a = 2$ and $b = c = 1/2$ (or any cyclic permutation).

\[\square\]

**P 1.30.** Let $a, b, c$ be nonnegative real numbers such that $a \geq b \geq c$ and $a + b + c = 3$. Prove that

$$
\frac{1}{a^2 + 3} + \frac{1}{b^2 + 3} + \frac{1}{c^2 + 3} \leq \frac{3}{4}.
$$

*(Vasile Cîrtoaje, 2005)*

**First Solution** (by Nguyen Van Quy). Write the inequality as follows:

$$
\left(\frac{1}{a^2 + 3} - \frac{3-a}{8}\right) + \left(\frac{1}{b^2 + 3} - \frac{3-b}{8}\right) \leq \left(\frac{3-c}{8} - \frac{1}{c^2 + 3}\right),
$$

$$
\frac{(a-1)^3}{a^2 + 3} + \frac{(b-1)^3}{b^2 + 3} \leq \frac{(1-c)^3}{c^2 + 3}.
$$

Indeed, we have

$$
\frac{(1-c)^3}{c^2 + 3} = \frac{(a-1+b-1)^3}{c^2 + 3} \geq \frac{(a-1)^3 + (b-1)^3}{c^2 + 3} \geq \frac{(a-1)^3}{a^2 + 3} + \frac{(b-1)^3}{b^2 + 3}.
$$

The proof is completed. The equality holds for $a = b = c = 1$.

**Second Solution.** Let $d$ be a positive number such that

$$
c + d = 2.
$$

We have

$$
a + b = 1 + d, \quad d \geq a \geq b \geq 1.
$$

In addition, we claim that

$$
\frac{1}{c^2 + 3} + \frac{1}{d^2 + 3} \leq \frac{1}{2}.
$$

Indeed,

$$
\frac{1}{2} - \frac{1}{c^2 + 3} - \frac{1}{d^2 + 3} = \frac{(cd-1)^2}{2(c^2 + 3)(d^2 + 3)} \geq 0.
$$

Thus, it suffices to show that

$$
\frac{1}{a^2 + 3} + \frac{1}{b^2 + 3} \leq \frac{1}{d^2 + 3} + \frac{1}{4}.
$$
Since
\[
\frac{1}{a^2 + 3} - \frac{1}{d^2 + 3} = \frac{(d-a)(d+a)}{(a^2 + 3)(d^2 + 3)}, \quad \frac{1}{4} - \frac{1}{b^2 + 3} = \frac{(b-1)(b+1)}{4(b^2 + 3)},
\]
we need to prove that
\[
\frac{d + a}{(a^2 + 3)(d^2 + 3)} \leq \frac{b + 1}{4(b^2 + 3)}.
\]
We can get this inequality by multiplying the inequalities
\[
\frac{d + a}{d^2 + 3} \leq \frac{a + 1}{4},
\]
\[
a + 1 \leq \frac{b + 1}{b^2 + 3}.
\]
We have
\[
\frac{a + 1}{4} - \frac{d + a}{d^2 + 3} = \frac{(d-1)(ad + a + d - 3)}{4(d^2 + 3)} \geq 0,
\]
\[
b + 1 - \frac{a + 1}{b^2 + 3} = \frac{(a-b)(ab + a + b - 3)}{(a^2 + 3)(b^2 + 3)} \geq 0.
\]

\[\square\]

**P 1.31.** Let \(a, b, c\) be nonnegative real numbers such that \(a + b + c = 3\). Prove that
\[
\frac{1}{2a^2 + 3} + \frac{1}{2b^2 + 3} + \frac{1}{2c^2 + 3} \geq \frac{3}{5}.
\]

*(Vasile Cîrtoaje, 2005)*

**First Solution (by Nguyen Van Quy).** Write the inequality as
\[
\sum \left(\frac{1}{3} - \frac{1}{2a^2 + 3}\right) \leq \frac{2}{5},
\]
\[
\sum \frac{a^2}{2a^2 + 5} \leq \frac{3}{5}.
\]
Using the Cauchy-Schwarz inequality gives
\[
\frac{25}{3(2a^2 + 3)} = \frac{25}{6a^2 + (a + b + c)^2}
\]
\[
= \frac{(2 + 2 + 1)^2}{2(2a^2 + bc) + 2a(a + b + c) + a^2 + b^2 + c^2}
\]
\[
\leq \frac{2^2}{2(2a^2 + bc)} + \frac{2^2}{2a(a + b + c)} + \frac{1}{a^2 + b^2 + c^2}.
\]
hence
\[\sum \frac{25a^2}{3(2a^2 + 3)} \leq \sum \frac{2a^2}{2a^2 + bc} + \sum \frac{2a}{a + b + c} + \sum \frac{a^2}{a^2 + b^2 + c^2} = \sum \frac{2a^2}{2a^2 + bc} + 3.\]

Therefore, it suffices to show that
\[\sum \frac{a^2}{2a^2 + bc} \leq 1,\]
which is equivalent to
\[\sum \left(\frac{1}{2} - \frac{a^2}{2a^2 + bc}\right) \geq \frac{1}{2},\]
\[\sum \frac{bc}{2a^2 + bc} \geq 1.\]

Using again the Cauchy-Schwarz inequality, we get
\[\sum \frac{bc}{2a^2 + bc} \geq \frac{(\sum bc)^2}{\sum bc(2a^2 + bc)} = 1.\]

The equality holds for \(a = b = c = 1\), as well as for \(a = 0\) and \(b = c = 3/2\) (or any cyclic permutation).

**Second Solution.** First, we can check that the desired inequality becomes an equality for \(a = b = c = 1\), and for \(a = 0\) and \(b = c = 3/2\). Consider then the inequality \(f(x) \geq 0\), where
\[f(x) = \frac{1}{2x^2 + 3} - A - Bx.\]

We have
\[f'(x) = \frac{-4x}{(2x^2 + 3)^2} - B.\]

From the conditions \(f(1) = 0\) and \(f'(1) = 0\), we get \(A = 9/25\) and \(B = -4/25\). Also, from the conditions \(f(3/2) = 0\) and \(f'(3/2) = 0\), we get \(A = 22/75\) and \(B = -8/75\). From these values of \(A\) and \(B\), we obtain the identities
\[\frac{1}{2x^2 + 3} - \frac{9 - 4x}{25} = \frac{2(x - 1)^2(4x - 1)}{25(2x^2 + 3)},\]
\[\frac{1}{2x^2 + 3} - \frac{22 - 8x}{75} = \frac{(2x - 3)^2(4x + 1)}{75(2x^2 + 3)},\]
and the inequalities
\[\frac{1}{2x^2 + 3} \geq \frac{9 - 4x}{25}, \quad x \geq \frac{1}{4},\]
Without loss of generality, assume that $a \geq b \geq c$.

Case 1: $a \geq b \geq c \geq \frac{1}{4}$. By summing the inequalities

$$\frac{1}{2a^2 + 3} \geq \frac{9 - 4a}{25}, \quad \frac{1}{2b^2 + 3} \geq \frac{9 - 4b}{25}, \quad \frac{1}{2c^2 + 3} \geq \frac{9 - 4c}{25},$$

we get

$$\frac{1}{2a^2 + 3} + \frac{1}{2b^2 + 3} + \frac{1}{2c^2 + 3} \geq \frac{27 - 4(a + b + c)}{25} = \frac{3}{5}.$$

Case 2: $a \geq b \geq \frac{1}{4} \geq c$. We have

$$\sum \frac{1}{2a^2 + 3} \geq \frac{22 - 8a}{75} + \frac{22 - 8b}{75} + \frac{1}{2c^2 + 3} = \frac{44 - 8(a + b)}{75} + \frac{1}{2c^2 + 3} = \frac{20 + 8c}{75} + \frac{1}{2c^2 + 3}.$$

Therefore, it suffices to show that

$$\frac{20 + 8c}{75} + \frac{1}{2c^2 + 3} \geq \frac{3}{5},$$

which is equivalent to the obvious inequality

$$c(8c^2 - 25c + 12) \geq 0.$$

Case 3: $a \geq \frac{1}{4} \geq b \geq c$. We have

$$\sum \frac{1}{2a^2 + 3} > \frac{1}{2b^2 + 3} + \frac{1}{2c^2 + 3} \geq \frac{2}{8 + 3} > \frac{3}{5}.$$
First Solution. Let \( b_1 \) and \( c_1 \) be positive numbers such that
\[
b + b_1 = 2, \quad c + c_1 = 2.
\]
We have
\[
b_1 + c_1 = 1 + a, \quad 1 \leq b_1 \leq c_1 \leq a.
\]
In addition, we claim that
\[
\frac{1}{b^2 + 2} + \frac{1}{b_1^2 + 2} \geq \frac{2}{3}, \quad \frac{1}{c^2 + 2} + \frac{1}{c_1^2 + 2} \geq \frac{2}{3}.
\]
Indeed,
\[
\frac{1}{b^2 + 2} + \frac{1}{b_1^2 + 2} = \frac{2}{3(b^2 + 2)(b_1^2 + 2)} = \frac{bb_1(b - b_1)^2}{6(b^2 + 2)(b_1^2 + 2)} \geq 0,
\]
\[
\frac{1}{c^2 + 2} + \frac{1}{c_1^2 + 2} = \frac{cc_1(c - c_1)^2}{6(c^2 + 2)(c_1^2 + 2)} \geq 0.
\]
Using these inequalities, it suffices to show that
\[
\frac{1}{a^2 + 2} + \frac{1}{3} \geq \frac{1}{b_1^2 + 2} + \frac{1}{c_1^2 + 2}.
\]
Since
\[
\frac{1}{3} \frac{1}{b_1^2 + 2} = \frac{(b_1 - 1)(b_1 + 1)}{3(b_1^2 + 2)}, \quad \frac{1}{c_1^2 + 2} - \frac{1}{a^2 + 2} = \frac{(a - c_1)(a + c_1)}{(a^2 + 2)(c_1^2 + 2)},
\]
we need to prove that
\[
\frac{b_1 + 1}{3(b_1^2 + 2)} \geq \frac{a + c_1}{(a^2 + 2)(c_1^2 + 2)}.
\]
We can get this inequality by multiplying the inequalities
\[
\frac{b_1 + 1}{b_1^2 + 2} \geq \frac{c_1 + 1}{c_1^2 + 2}, \quad \frac{c_1 + 1}{3} \geq \frac{a + c_1}{a^2 + 2}.
\]
We have
\[
\frac{b_1 + 1}{b_1^2 + 2} - \frac{c_1 + 1}{c_1^2 + 2} = \frac{(c_1 - b_1)(b_1c_1 + b_1 + c_1 - 2)}{(b_1^2 + 2)(c_1^2 + 2)} \geq 0,
\]
\[
\frac{c_1 + 1}{3} - \frac{a + c_1}{a^2 + 2} = \frac{(a - 1)(ac_1 + a + c_1 - 2)}{3(a^2 + 2)} \geq 0.
\]
The proof is completed. The equality holds for $a = b = c = 1$, as well as for $a = 2, b = 1$ and $c = 0$.

**Second Solution.** First, we can check that the desired inequality becomes an equality for $a = b = c = 1$, and also for $a = 2, b = 1, c = 0$. Consider then the inequality $f(x) \geq 0$, where

$$f(x) = \frac{1}{x^2 + 2} - A - Bx.$$ 

We have

$$f'(x) = \frac{-2x}{(x^2 + 2)^2} - B.$$ 

From the conditions $f(1) = 0$ and $f'(1) = 0$, we get $A = 5/9$ and $B = -2/9$. Also, from the conditions $f(2) = 0$ and $f'(2) = 0$, we get $A = 7/18$ and $B = -1/9$. From these values of $A$ and $B$, we obtain the identities

$$\frac{1}{x^2 + 2} - \frac{5 - 2x}{9} = \frac{(x - 1)^2(2x - 1)}{9(x^2 + 2)},$$

$$\frac{1}{x^2 + 2} - \frac{7 - 2x}{18} = \frac{(x - 2)^2(2x + 1)}{18(x^2 + 2)},$$

and the inequalities

$$\frac{1}{x^2 + 2} \geq \frac{5 - 2x}{9}, \quad x \geq \frac{1}{2},$$

$$\frac{1}{x^2 + 2} \geq \frac{7 - 2x}{18}, \quad x \geq 0.$$ 

Let us define

$$g(x) = \frac{1}{x^2 + 2}.$$ 

Notice that for $d \in (0, \sqrt{2}]$ and $x \in [0, d]$, we have

$$g(x) \geq g(0) + \frac{g(d) - g(0)}{d} x,$$

because

$$g(x) - g(0) - \frac{g(d) - g(0)}{d} x = \frac{x(d - x)(2 - dx)}{2(d^2 + 2)(x^2 + 2)} \geq 0.$$ 

For $d = 1/2$ and $d = 1$, we get the inequalities

$$\frac{1}{x^2 + 2} \geq \frac{9 - 2x}{18}, \quad 0 \leq x \leq \frac{1}{2},$$

$$\frac{1}{x^2 + 2} \geq \frac{3 - x}{6}, \quad 0 \leq x \leq 1.$$ 

Consider further two cases: $c \geq 1/2$ and $c \leq 1/2.$
Case 1: \( c \geq \frac{1}{2} \). By summing the inequalities
\[
\frac{1}{a^2 + 2} \geq \frac{5 - 2a}{9}, \quad \frac{1}{b^2 + 2} \geq \frac{5 - 2b}{9}, \quad \frac{1}{c^2 + 2} \geq \frac{5 - 2c}{9},
\]
we get
\[
\frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2} \geq \frac{15 - 2(a + b + c)}{9} = 1.
\]

Case 2: \( c \leq \frac{1}{2} \). We have
\[
\frac{1}{a^2 + 2} \geq \frac{7 - 2a}{18},
\]
\[
\frac{1}{b^2 + 2} \geq \frac{3 - b}{6} \geq \frac{8 - 2b}{18},
\]
\[
\frac{1}{c^2 + 2} \geq \frac{9 - 2c}{18}.
\]
Therefore,
\[
\frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2} \geq \frac{7 - 2a}{18} + \frac{8 - 2b}{18} + \frac{9 - 2c}{18} = 1.
\]

\( \Box \)

**P 1.33.** Let \( a, b, c \) be nonnegative real numbers such that \( ab + bc + ca = 3 \). Prove that
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{a + b + c}{6} + \frac{3}{a + b + c}.
\]

(Vasile Cîrtoaje, 2007)

**First Solution.** Denoting \( x = a + b + c \), we have
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} = \frac{(a + b + c)^2 + ab + bc + ca}{(a + b + c)(ab + bc + ca) - abc} = \frac{x^2 + 3}{3x - abc}.
\]

Then, the inequality becomes
\[
\frac{x^2 + 3}{3x - abc} \geq \frac{x}{6} + \frac{3}{x},
\]
or
\[
3(x^3 + 9abc - 12x) + abc(x^2 - 9) \geq 0.
\]
Indeed, which is equivalent to

\[ \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{a+b+c}{2(ab+bc+ca)} + \frac{3}{a+b+c}, \]

we get \( x^3 + 9abc - 12 \geq 0 \). The equality holds for \( a = b = c = 1 \), and for \( a = 0 \) and \( b = c = \sqrt{3} \) (or any cyclic permutation).

**Second Solution.** We apply the SOS method. Write the inequality as follows:

\[
2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{(a+b+c)^2}{ab+bc+ca} + 6,
\]

\[
[(a+b) + (b+c) + (c+a)]\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) - 9 \geq \frac{(a+b+c)^2}{ab+bc+ca} - 3,
\]

\[
\sum \frac{(b-c)^2}{(a+b)(c+a)} \geq \frac{1}{2(ab+bc+ca)} \sum (b-c)^2,
\]

\[
\sum \frac{ab+bc+ca-a^2}{(a+b)(c+a)}(b-c)^2 \geq 0.
\]

Without loss of generality, assume that \( a \geq b \geq c \). Since \( ab+bc+ca-c^2 \geq 0 \), it suffices to show that

\[
\frac{ab+bc+ca-a^2}{(a+b)(c+a)}(b-c)^2 + \frac{ab+bc+ca-b^2}{(b+c)(a+b)}(c-a)^2 \geq 0.
\]

Since \( ab+bc+ca-b^2 \geq 0 \) and \( c-a)^2 \geq (b-c)^2 \), it is enough to prove that

\[
\frac{ab+bc+ca-a^2}{(a+b)(c+a)}(b-c)^2 + \frac{ab+bc+ca-b^2}{(b+c)(a+b)}(b-c)^2 \geq 0.
\]

This is true if

\[
\frac{ab+bc+ca-a^2}{(a+b)(c+a)} + \frac{ab+bc+ca-b^2}{(b+c)(a+b)} \geq 0,
\]

which is equivalent to

\[
\frac{3-a^2}{3+a^2} + \frac{3-b^2}{3+b^2} \geq 0,
\]

Indeed,

\[
\frac{3-a^2}{3+a^2} + \frac{3-b^2}{3+b^2} = \frac{2(9-a^2b^2)}{(3+a^2)(3+b^2)} = \frac{2c(a+b)(3+ab)}{(3+a^2)(3+b^2)} \geq 0.
\]

\( \square \)
P 1.34. Let \( a, b, c \) be nonnegative real numbers such that \( ab + bc + ca = 3 \). Prove that

\[
\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} \geq \frac{3}{2}.
\]

(Vasile Cîrtoaje, 2005)

**First Solution.** After expanding, the inequality can be restated as

\[
a^2 + b^2 + c^2 + 3 \geq a^2 b^2 + b^2 c^2 + c^2 a^2 + 3a^2 b^2 c^2.
\]

From

\[
(a + b + c)(ab + bc + ca) - 9abc = a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \geq 0,
\]

we get

\[
a + b + c \geq 3abc.
\]

So, it suffices to show that

\[
a^2 + b^2 + c^2 + 3 \geq a^2 b^2 + b^2 c^2 + c^2 a^2 + abc(a + b + c).
\]

This is equivalent to the homogeneous inequalities

\[
(ab + bc + ca)(a^2 + b^2 + c^2) + (ab + bc + ca)^2 \geq 3(a^2 b^2 + b^2 c^2 + c^2 a^2) + 3abc(a + b + c),
\]

\[
ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \geq 2(a^2 b^2 + b^2 c^2 + c^2 a^2),
\]

\[
ab(a - b)^2 + bc(b - c)^2 + ca(c - a)^2 \geq 0.
\]

The equality holds for \( a = b = c = 1 \), and for \( a = 0 \) and \( b = c = \sqrt{3} \) (or any cyclic permutation).

**Second Solution.** Without loss of generality, assume that \( a = \min\{a, b, c\} \). From \( ab + bc + ca = 3 \), we get \( bc \geq 1 \). Also, from

\[
(a + b + c)(ab + bc + ca) - 9abc = a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \geq 0,
\]

we get

\[
a + b + c \geq 3abc.
\]

The desired inequality follows by summing the inequalities

\[
\frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} \geq \frac{2}{bc + 1},
\]

\[
\frac{1}{a^2 + 1} + \frac{2}{bc + 1} \geq \frac{3}{2}.
\]
We have
\[
\frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} - \frac{2}{bc + 1} = \frac{b(c - b)}{(b^2 + 1)(bc + 1)} + \frac{c(b - c)}{(c^2 + 1)(bc + 1)}
\]
\[= \frac{(b - c)^2(bc - 1)}{(b + 1)(c^2 + 1)(bc + 1)} \geq 0\]
and
\[
\frac{1}{a^2 + 1} + \frac{2}{bc + 1} - \frac{3}{2} = \frac{a^2 - bc + 3 - 3a^2bc}{2(a^2 + 1)(bc + 1)} = \frac{a(a + b + c - 3abc)}{2(a^2 + 1)(bc + 1)} \geq 0.
\]

**Third Solution.** Since
\[
\frac{1}{a^2 + 1} = 1 - \frac{a^2}{a^2 + 1}, \quad \frac{1}{b^2 + 1} = 1 - \frac{b^2}{b^2 + 1}, \quad \frac{1}{c^2 + 1} = 1 - \frac{c^2}{c^2 + 1},
\]
we can rewrite the inequality as
\[
\frac{a^2}{a^2 + 1} + \frac{b^2}{b^2 + 1} + \frac{c^2}{c^2 + 1} \leq \frac{3}{2},
\]
or, in the homogeneous form
\[
\sum \frac{a^2}{3a^2 + ab + bc + ca} \leq \frac{1}{2}.
\]
According to the Cauchy-Schwarz inequality, we have
\[
\frac{4a^2}{3a^2 + ab + bc + ca} = \frac{(a + a)^2}{a(a + b + c) + (2a^2 + bc)} \leq \frac{a}{a + b + c} + \frac{a^2}{2a^2 + bc}.
\]
Then,
\[
\sum \frac{4a^2}{3a^2 + ab + bc + ca} \leq 1 + \sum \frac{a^2}{2a^2 + bc},
\]
and it suffices to show that
\[
\sum \frac{a^2}{2a^2 + bc} \leq 1,
\]
or, equivalently,
\[
\sum \frac{bc}{2a^2 + bc} \geq 1.
\]
This follows from the Cauchy-Schwarz inequality as follows:
\[
\sum \frac{bc}{2a^2 + bc} \geq \frac{(\sum bc)^2}{\sum bc(2a^2 + bc)} = \frac{\sum b^2c^2 + 2abc \sum a}{2abc \sum a + \sum b^2c^2} = 1.
\]
Remark. We can write the inequality in P 1.34 in the homogeneous form
\[
\frac{1}{1 + \frac{3a^2}{ab + bc + ca}} + \frac{1}{1 + \frac{3b^2}{ab + bc + ca}} + \frac{1}{1 + \frac{3c^2}{ab + bc + ca}} \geq \frac{3}{2}.
\]
Substituting \(a, b, c\) by \(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\), respectively, we get
\[
\frac{x}{x + \frac{3yz}{x + y + z}} + \frac{y}{y + \frac{3zx}{x + y + z}} + \frac{z}{z + \frac{3xy}{x + y + z}} \geq \frac{3}{2}.
\]
So, we find the following result.

- If \(x, y, z\) are positive real numbers such that \(x + y + z = 3\), then
  \[
  \frac{x}{x + yz} + \frac{y}{y + zx} + \frac{z}{z + xy} \geq \frac{3}{2}.
  \]

\(\square\)

P 1.35. Let \(a, b, c\) be positive real numbers such that \(ab + bc + ca = 3\). Prove that
\[
\frac{a^2}{a^2 + b + c} + \frac{b^2}{b^2 + c + a} + \frac{c^2}{c^2 + a + b} \geq 1.
\]

\((\text{Vasile Cîrtoaje}, 2005)\)

Solution. We apply the Cauchy-Schwarz inequality in the following way
\[
\sum \frac{a^2}{a^2 + b + c} \geq \frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{\sum a(a^2 + b + c)} = \frac{\sum a^3 + 2\sum(ab)^{3/2}}{\sum a^3 + 6}.
\]
Then, we still have to show that
\[
(ab)^{3/2} + (bc)^{3/2} + (ca)^{3/2} \geq 3.
\]
By the AM-GM inequality,
\[
(ab)^{3/2} = \frac{(ab)^{3/2} + (ab)^{3/2} + 1}{2} \geq \frac{3ab}{2} - \frac{1}{2},
\]
and hence
\[
\sum (ab)^{3/2} \geq \frac{3}{2} \sum ab - \frac{3}{2} = 3.
\]
The equality holds for \(a = b = c = 1\).

\(\square\)
P 1.36. Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 3$. Prove that

\[
\frac{bc + 4}{a^2 + 4} + \frac{ca + 4}{b^2 + 4} + \frac{ab + 4}{c^2 + 4} \leq 3 \leq \frac{bc + 2}{a^2 + 2} + \frac{ca + 2}{b^2 + 2} + \frac{ab + 2}{c^2 + 2}.
\]

(Vasile Cîrtoaje, 2007)

Solution. More general, using the SOS method, we will show that

\[
(k - 3) \left( \frac{bc + k}{a^2 + k} + \frac{ca + k}{b^2 + k} + \frac{ab + k}{c^2 + k} - 3 \right) \leq 0
\]

for $k > 0$. This inequality is equivalent to

\[
(k - 3) \sum \frac{a^2 - bc}{a^2 + k} \geq 0.
\]

Since

\[
2 \sum \frac{a^2 - bc}{a^2 + k} = \sum \frac{(a - b)(a + c) + (a - c)(a + b)}{a^2 + k}
= \sum \frac{(a - b)(a + c)}{a^2 + k} + \sum \frac{(b - a)(b + c)}{b^2 + k}
= (k - ab - bc - ca) \sum \frac{(a - b)^2}{(a^2 + p)(b^2 + p)}
= (k - 3) \sum \frac{(a - b)^2}{(a^2 + p)(b^2 + p)},
\]

we have

\[
2(k - 3) \sum \frac{a^2 - bc}{a^2 + k} = (k - 3)^2 \sum \frac{(a - b)^2}{(a^2 + k)(b^2 + k)} \geq 0.
\]

Equality in both inequalities holds for $a = b = c = 1$. \qed

P 1.37. Let $a, b, c$ be nonnegative real numbers such that $ab + bc + ca = 3$. If $k \geq 2 + \sqrt{3}$, then

\[
\frac{1}{a + k} + \frac{1}{b + k} + \frac{1}{c + k} \leq \frac{3}{1 + k}.
\]

(Vasile Cîrtoaje, 2007)

Solution. Let us denote $p = a + b + c$, $p \geq 3$. By expanding, the inequality becomes

\[
k(k - 2)p + 3abc \geq 3(k - 1)^2.
\]
Since this inequality is true for \( p \geq \frac{3(k-1)^2}{(k^2-2k)} \), consider further that
\[
p \leq \frac{3(k-1)^2}{k(k-2)}.
\]
From Schur’s inequality
\[
(a + b + c)^3 + 9abc \geq 4(ab + bc + ca)(a + b + c),
\]
we get \( 9abc \geq 12p - p^3 \). Therefore, it suffices to prove that
\[
3k(k-2)p + 12p - p^3 \geq 9(k-1)^2,
\]
or, equivalently,
\[
(p - 3)[(3(k-1)^2 - p^2 - 3p] \geq 0.
\]
Thus, it remains to prove that
\[
3(k-1)^2 - p^2 - 3p \geq 0.
\]
Since \( p \leq \frac{3(k-1)^2}{(k^2-2k)} \) and \( k \geq 2 + \sqrt{3} \), we have
\[
3(k-1)^2 - p^2 - 3p \geq 3(k-1)^2 - \frac{9(k-1)^4}{k^2(k-2)^2} - \frac{9(k-1)^2}{k(k-2)}
\]
\[
= \frac{3(k-1)^2(k^2 - 3)(k^2 - 4k + 1)}{k^2(k-2)^2} \geq 0.
\]
The equality holds for \( a = b = c = 1 \). In the case \( k = 2 + \sqrt{3} \), the equality holds again for \( a = 0 \) and \( b = c = \sqrt{3} \) (or any cyclic permutation).

\( \square \)

**P 1.38.** Let \( a, b, c \) be nonnegative real numbers such that \( a^2 + b^2 + c^2 = 3 \). Prove that
\[
\frac{a(b + c)}{1 + bc} + \frac{b(c + a)}{1 + ca} + \frac{c(a + b)}{1 + ab} \leq 3.
\]

*(Vasile Cîrtoaje, 2010)*

**Solution.** Write the inequality in the homogeneous form
\[
\sum \frac{a(b + c)}{a^2 + b^2 + c^2 + 3bc} \leq 1,
\]
or
\[
\sum \left[ \frac{a(b + c)}{a^2 + b^2 + c^2 + 3bc} - \frac{a}{a + b + c} \right] \leq 0,
\]
\[ \sum \frac{a(a-b)(a-c)}{a^2 + b^2 + c^2 + 3bc} \geq 0. \]

Without loss of generality, assume that \( a \geq b \geq c \). Then, it suffices to prove that

\[ \frac{a(a-b)(a-c)}{a^2 + b^2 + c^2 + 3bc} + \frac{b(b-c)(b-a)}{a^2 + b^2 + c^2 + 3ca} \geq 0. \]

This is true if

\[ \frac{a(a-c)}{a^2 + b^2 + c^2 + 3bc} \geq \frac{b(b-c)}{a^2 + b^2 + c^2 + 3ca}. \]

Since \( a(a-c) \geq b(b-c) \) and

\[ \frac{1}{a^2 + b^2 + c^2 + 3bc} \geq \frac{1}{a^2 + b^2 + c^2 + 3ca}, \]

the conclusion follows. The equality holds for \( a = b = c = 1 \). and for \( a = 0 \) and \( b = c = \sqrt{3}/2 \) (or any cyclic permutation).

\[ \square \]

**P 1.39.** Let \( a, b, c \) be positive real numbers such that \( a^2 + b^2 + c^2 = 3 \). Prove that

\[ \frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} \leq 3. \]

*(Cezar Lupu, 2005)*

**First Solution.** We apply the SOS method. Write the inequality in the homogeneous form

\[ \sum \left( \frac{b^2 + c^2}{b + c} - \frac{b + c}{2} \right) \geq \sqrt{3(a^2 + b^2 + c^2)} - a - b - c, \]

or

\[ \sum \frac{(b-c)^2}{2(b+c)} \geq \frac{\sum(b-c)^2}{\sqrt{3(a^2 + b^2 + c^2)} + a + b + c}. \]

Since \( \sqrt{3(a^2 + b^2 + c^2)} + a + b + c \geq 2(a + b + c) > 2(b + c) \), the conclusion follows. The equality holds for \( a = b = c = 1 \).

**Second Solution.** By virtue of the Cauchy-Schwarz inequality, we have

\[ \sum \frac{a^2 + b^2}{a + b} \geq \frac{(\sum a^2 + b^2)^2}{\sum(a + b)} = \frac{2 \sum a^2 + 2 \sum \sqrt{(a^2 + b^2)(a^2 + c^2)}}{2 \sum a} \]

\[ \geq \frac{2 \sum a^2 + 2 \sum (a^2 + bc)}{2 \sum a} = \frac{3 \sum a^2 + (\sum a)^2}{2 \sum a} \]

\[ = \frac{9 + (\sum a)^2}{2 \sum a} = 3 + \frac{(\sum a - 3)^2}{2 \sum a} \geq 3. \]

\[ \square \]
P 1.40. Let \( a, b, c \) be positive real numbers such that \( a^2 + b^2 + c^2 = 3 \). Prove that

\[
\frac{ab}{a + b} + \frac{bc}{b + c} + \frac{ca}{c + a} + 2 \leq \frac{7}{6}(a + b + c).
\]

(Vasile Cîrtoaje, 2011)

Solution. We apply the SOS method. Write the inequality as

\[
3 \sum \left( b + c - \frac{4bc}{b + c} \right) \geq 8(3 - a - b - c).
\]

Since

\[
b + c - \frac{4bc}{b + c} = \frac{(b - c)^2}{b + c}
\]

and

\[
3 - a - b - c = \frac{9 - (a + b + c)^2}{3 + a + b + c} = \frac{3(a^2 + b^2 + c^2) - (a + b + c)^2}{3 + a + b + c}
\]

we can write the inequality as

\[
S_a(b - c)^2 + S_b(c - a)^2 + S_c(a - b)^2 \geq 0,
\]

where

\[
S_a = \frac{3}{b + c} - \frac{8}{3 + a + b + c}.
\]

Without loss of generality, assume that \( a \geq b \geq c \). Since \( S_a \geq S_b \geq S_c \), it suffices to show that \( S_b + S_c \geq 0 \). Indeed, if this condition is fulfilled, then \( S_a \geq S_b \geq (S_b + S_c)/2 \geq 0 \), and hence

\[
S_a(b - c)^2 + S_b(c - a)^2 + S_c(a - b)^2 \geq S_b(c - a)^2 + S_c(a - b)^2
\]

\[
\geq S_b(a - b)^2 + S_c(a - b)^2 = (a - b)^2(S_b + S_c) \geq 0.
\]

Since

\[
S_b + S_c = \frac{4(9 - 5a - b - c)}{(a + b + 2c)(3 + a + b + c)},
\]

we need to show that \( 9 \geq 5a + b + c \). This follows immediately from the Cauchy-Schwarz inequality

\[
(25 + 1 + 1)(a^2 + b^2 + c^2) \geq (5a + b + c)^2.
\]

Thus, the proof is completed. The equality holds for \( a = b = c = 1 \), and for \( a = 5/3 \) and \( b = c = 1/3 \) (or any cyclic permutation). 

\[ \square \]
P 1.41. Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 = 3\). Prove that

\[
\begin{align*}
(a) & \quad \frac{1}{3 - ab} + \frac{1}{3 - bc} + \frac{1}{3 - ca} \leq \frac{3}{2}; \\
(b) & \quad \frac{1}{\sqrt{6} - ab} + \frac{1}{\sqrt{6} - bc} + \frac{1}{\sqrt{6} - ca} \leq \frac{3}{\sqrt{6} - 1}.
\end{align*}
\]

(Vasile Cîrtoaje, 2005)

Solution. (a) Since

\[
\frac{3}{3 - ab} = 1 + \frac{ab}{3 - ab} = 1 + \frac{2ab}{a^2 + b^2 + 2c^2 + (a - b)^2} \leq 1 + \frac{2ab}{a^2 + b^2 + 2c^2} \leq 1 + \frac{(a + b)^2}{2(a^2 + b^2 + 2c^2)},
\]

it suffices to prove that

\[
\frac{(a + b)^2}{a^2 + b^2 + 2c^2} + \frac{(b + c)^2}{b^2 + c^2 + 2a^2} + \frac{(c + a)^2}{c^2 + a^2 + 2b^2} \leq 3.
\]

By the Cauchy-Schwarz inequality, we have

\[
\frac{(a + b)^2}{a^2 + b^2 + 2c^2} = \frac{(a + b)^2}{(a^2 + c^2) + (b^2 + c^2)} \leq \frac{a^2}{a^2 + c^2} + \frac{b^2}{b^2 + c^2}.
\]

Thus,

\[
\sum \frac{(a + b)^2}{a^2 + b^2 + 2c^2} \leq \sum \frac{a^2}{a^2 + c^2} + \sum \frac{b^2}{b^2 + c^2} = \sum \frac{a^2}{a^2 + c^2} + \sum \frac{c^2}{c^2 + a^2} = 3.
\]

The equality holds for \(a = b = c\).

(b) According to P 1.28, the following inequality holds

\[
\frac{1}{6 - a^2 b^2} + \frac{1}{6 - b^2 c^2} + \frac{1}{6 - c^2 a^2} \leq \frac{3}{5}.
\]

Since

\[
\frac{2\sqrt{6}}{6 - a^2 b^2} = \frac{1}{\sqrt{6} - ab} + \frac{1}{\sqrt{6} + ab},
\]

this inequality becomes

\[
\sum \frac{1}{\sqrt{6} - ab} + \sum \frac{1}{\sqrt{6} + ab} \leq \frac{6\sqrt{6}}{5}.
\]
Thus, it suffices to show that
\[ \sum \frac{1}{\sqrt{6} + ab} \geq \frac{3}{\sqrt{6} + 1}. \]

Since \( ab + bc + ca \leq a^2 + b^2 + c^2 = 3 \), by the Cauchy-Schwarz inequality, we have
\[ \sum \frac{1}{\sqrt{6} + ab} \geq \frac{9}{3\sqrt{6} + ab + bc + ca} \geq \frac{3}{\sqrt{6} + 1}. \]
The equality holds for \( a = b = c = 1 \).

\[ \square \]

**P 1.42.** Let \( a, b, c \) be positive real numbers such that \( a^2 + b^2 + c^2 = 3 \). Prove that
\[ \frac{1}{1 + a^5} + \frac{1}{1 + b^5} + \frac{1}{1 + c^5} \geq \frac{3}{2}. \]

*(Vasile Cîrtoaje, 2007)*

**Solution.** Let \( a = \min\{a, b, c\} \). There are two cases to consider

**Case 1:** \( a \geq 1/2 \). The desired inequality follows by summing the inequalities
\[ \frac{8}{1 + a^5} \geq 9 - 5a^2, \quad \frac{8}{1 + b^5} \geq 9 - 5b^2, \quad \frac{8}{1 + c^5} \geq 9 - 5c^2; \]

To obtain these inequalities, we start from the inequality \( \frac{8}{1 + x^5} \geq p + qx^2 \), whose coefficients \( p \) and \( q \) will be determined such that the polynomial \( P(x) = 8 - (1 + x^5)(p + qx^2) \) divides by \( (x - 1)^2 \). It is easy to check that \( P(1) = 0 \) involves \( p + q = 4 \), when
\[ P(x) = 4(2 - x^2 - x^7) - p(1 - x^2 + x^5 - x^7) = (1 - x)Q(x), \]
where
\[ Q(x) = 4(2 + 2x + x^2 + x^3 + x^4 + x^5 + x^6) - p(1 + x + x^5 + x^6). \]

In addition, \( Q(1) = 0 \) involves \( p = 9 \). In this case,
\[ P(x) = (1 - x)^2(5x^5 + 10x^4 + 6x^3 + 2x^2 - 2x - 1) \]
\[ = (1 - x)^2[x^5 + (2x - 1)(2x^4 + 6x^3 + 6x^2 + 4x + 1)] \geq 0. \]

**Case 2:** \( a \geq 1/2 \). Write the inequality as
\[ \frac{1}{1 + a^5} - \frac{1}{2} \geq \frac{b^5c^5 - 1}{(1 + b^5)(1 + c^5)}. \]
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Since
\[
\frac{1}{1 + a^5} - \frac{1}{2} \geq \frac{32}{33} - \frac{1}{2} = \frac{31}{66}
\]
and
\[
(1 + b^5)(1 + c^5) \geq (1 + \sqrt{b^5c^5})^2,
\]
it suffices to show that
\[
31(1 + \sqrt{b^5c^5})^2 \geq 66(b^5c^5 - 1),
\]
which is equivalent to \(bc \leq (97/35)^{2/5}\). Indeed, from
\[
3 = a^2 + b^2 + c^2 > b^2 + c^2 \geq 2bc,
\]
we get \(bc < 3/2 < (97/35)^{2/5}\). This completes the proof. The equality holds for \(a = b = c = 1\).

\[ \Box \]

**P 1.43.** Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that
\[
\frac{1}{a^2 + a + 1} + \frac{1}{b^2 + b + 1} + \frac{1}{c^2 + c + 1} \geq 1.
\]

**First Solution.** Using the substitutions \(a = yz/x^2\), \(b = zx/y^2\), \(c = xy/z^2\), where \(x, y, z\) are positive real numbers, the inequality becomes
\[
\sum \frac{x^4}{x^4 + x^2yz + y^2z^2} \geq 1.
\]

By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{x^4}{x^4 + x^2yz + y^2z^2} \geq \frac{(\sum x^2)^2}{\sum(x^4 + x^2yz + y^2z^2)} = \frac{\sum x^4 + 2\sum y^2z^2}{\sum x^4 + xyz\sum x + \sum y^2z^2}.
\]
Therefore, it suffices to show that
\[
\sum y^2z^2 \geq xyz\sum x,
\]
which is equivalent to \(\sum x^2(y - z)^2 \geq 0\). The equality holds for \(a = b = c = 1\).

**Second Solution.** Using the substitutions \(a = y/x\), \(b = z/y\), \(c = x/z\), where \(x, y, z > 0\), we need to prove that
\[
\frac{x^2}{x^2 + xy + y^2} + \frac{y^2}{y^2 + yz + z^2} + \frac{z^2}{z^2 + zx + z^2} \geq 1.
\]
Since
\[ \frac{x^2(y^2 + z^2 + xy + yz + zx)}{x^2 + xy + y^2} = \frac{x^2z(x + y + z)}{x^2 + xy + y^2}, \]
multiplying by \(x^2 + y^2 + z^2 + xy + yz + zx\), the inequality can be written as
\[ \sum \frac{x^2z}{x^2 + xy + y^2} \geq \frac{xy + yz + zx}{x + y + z}. \]

By the Cauchy-Schwarz inequality, we have
\[ \sum \frac{x^2z}{x^2 + xy + y^2} \geq \frac{(\sum xz)^2}{\sum (x^2 + xy + y^2)} = \frac{xy + yz + zx}{x + y + z}. \]

**Remark.** The inequality in P 1.43 is a particular case of the following more general inequality: (Vasile Cîrtoaje, 2009).

- Let \(a_1, a_2, \ldots, a_n\) (\(n \geq 3\)) be positive real numbers such that \(a_1a_2 \cdots a_n = 1\). If \(p\) and \(q\) are nonnegative real numbers satisfying \(p + q = n - 1\), then
  \[ \sum_{i=1}^{i=n} \frac{1}{1 + pa_i + qa_i^2} \geq 1. \]

\[ \square \]

**P 1.44.** Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that
\[ \frac{1}{a^2 - a + 1} + \frac{1}{b^2 - b + 1} + \frac{1}{c^2 - c + 1} \leq 3. \]

**First Solution.** Since
\[ \frac{1}{a^2 - a + 1} + \frac{1}{a^2 + a + 1} = \frac{2(a^2 + 1)}{a^4 + a^2 + 1} = 2 - \frac{2a^4}{a^4 + a^2 + 1}, \]
we can rewrite the inequality as
\[ \sum \frac{1}{a^2 + a + 1} + 2 \sum \frac{a^4}{a^4 + a^2 + 1} \geq 3. \]
Thus, it suffices to show that
\[ \sum \frac{1}{a^2 + a + 1} \geq 1 \]
and
\[ \sum \frac{a^4}{a^4 + a^2 + 1} \geq 1. \]
The first inequality is just the inequality in P 1.43, while the second follows from the first by substituting \(a, b, c\) with \(a^{-2}, b^{-2}, c^{-2}\), respectively. The equality holds for \(a = b = c = 1\).

**Second Solution.** Write the inequality as

\[
\sum\left(\frac{1}{a^2 - a + 1}\right) \geq 1,
\]

\[
\sum \frac{(2a - 1)^2}{a^2 - a + 1} \geq 3.
\]

Let \(p = a + b + c\) and \(q = ab + bc + ca\). By the Cauchy-Schwarz inequality, we have

\[
\sum \frac{(2a - 1)^2}{a^2 - a + 1} \geq \frac{(2\sum a - 3)^2}{\sum(a^2 - a + 1)} = \frac{(2p - 3)^2}{p^2 - 2q - p + 3}.
\]

Thus, it suffices to show that

\[
(2p - 3)^2 \geq 3(p^2 - 2q - p + 3),
\]

which is equivalent to

\[
p^2 + 6q - 9p \geq 0.
\]

From the known inequality

\[
(ab + bc + ca)^2 \geq 3abc(a + b + c),
\]

we get \(q^2 \geq 3p\). Using this inequality and the AM-GM inequality, we get

\[
p^2 + 6q = p^2 + 3q + 3q \geq 3\sqrt[3]{9p^2q^2} \geq 3\sqrt[3]{9p^2(3p)} = 9p.
\]

\[\square\]

**P 1.45.** Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that

\[
\frac{3 + a}{(1 + a)^2} + \frac{3 + b}{(1 + b)^2} + \frac{3 + c}{(1 + c)^2} \geq 3.
\]

**Solution.** From the identity

\[
\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} - \frac{1}{1 + ab} = \frac{ab(a - b)^2 + (1 - ab)^2}{(1 + a)^2(1 + b)^2(1 + ab)},
\]
we get the known inequality
\[
\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}.
\]

We can also prove this inequality using the Cauchy-Schwarz inequality, as follows
\[
\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} - \frac{1}{1+ab} \geq \frac{(b+a)^2}{b^2(1+a)^2 + a^2(1+b)^2} - \frac{1}{1+ab} = \frac{ab[a^2 + b^2 - 2(a+b) + 2]}{(1+ab)[b^2(1+a)^2 + a^2(1+b)^2]} \geq 0.
\]

In addition, this inequality can be obtained by summing the inequalities
\[
\frac{1}{(1+a)^2} \geq \frac{b}{(a+b)(1+ab)},
\]
\[
\frac{1}{(1+b)^2} \geq \frac{a}{(a+b)(1+ab)}.
\]
Thus, we have
\[
\sum \frac{3+a}{(1+a)^2} = \sum \frac{2}{(1+a)^2} + \sum \frac{1}{1+a} = \sum \left[ \frac{1}{(1+a)^2} + \frac{1}{1+a} \right] + \sum \frac{1}{1+c} \geq \sum \frac{1}{1+ab} + \sum \frac{ab}{1+ab} = 3.
\]

The equality holds for \(a = b = c = 1\). \(\square\)

**P 1.46.** Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that
\[
\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} \geq 1.
\]

*(Vasile Cirtoaje, 2008)*

**Solution.** Write the inequality as
\[
\left( \frac{7 - 6a}{2 + a^2} + 1 \right) + \left( \frac{7 - 6b}{2 + b^2} + 1 \right) + \left( \frac{7 - 6c}{2 + c^2} + 1 \right) \geq 4,
\]
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\[
\frac{(3 - a)^2}{2 + a^2} + \frac{(3 - b)^2}{2 + b^2} + \frac{(3 - c)^2}{2 + c^2} \geq 4.
\]

Substituting \(a, b, c\) by \(1/a, 1/b, 1/c\), respectively, we need to prove that \(abc = 1\) involves

\[
\frac{(3a - 1)^2}{2a^2 + 1} + \frac{(3b - 1)^2}{2b^2 + 1} + \frac{(3c - 1)^2}{2c^2 + 1} \geq 4.
\]

By the Cauchy-Schwarz inequality, we have

\[
\sum \frac{(3a - 1)^2}{2a^2 + 1} \geq \left( \frac{3 \sum a - 3)^2}{\sum(2a^2 + 1)} \right) = \frac{9 \sum a^2 + 18 \sum ab - 18 \sum a + 9}{2 \sum a^2 + 3}.
\]

Thus, it suffices to prove that

\[
f(a) + f(b) + f(c) \geq 3,
\]

where

\[
f(x) = x^2 + 18 \left( \frac{1}{x} - x \right).
\]

In order to do this, we use the mixing method. Without loss of generality, assume that \(a = \max\{a, b, c\}, a \geq 1, bc \leq 1\). Since

\[
f(b) + f(c) - 2f(\sqrt{bc}) = (b - c)^2 + 18(\sqrt{b} - \sqrt{c})^2 \left( \frac{1}{bc} - 1 \right) \geq 0,
\]

it suffices to show that

\[
f(a) + 2f(\sqrt{bc}) \geq 3.
\]

Write this inequality as

\[
f(x^2) + 2f \left( \frac{1}{x} \right) \geq 3,
\]

where \(x = \sqrt{a}\). It is equivalent to

\[
x^6 - 18x^4 + 36x^3 - 3x^2 - 36x + 20 \geq 0,
\]

\[
(x - 1)^2(x - 2)^2(x + 1)(x + 5) \geq 0.
\]

Since the last inequality is true, the proof is completed. The equality holds for \(a = b = c = 1\), and also for \(a = 1/4\) and \(b = c = 2\) (or any cyclic permutation).

\[
\square
\]

\textbf{P 1.47.} Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that

\[
\frac{a^6}{1 + 2a^5} + \frac{b^6}{1 + 2b^5} + \frac{c^6}{1 + 2c^5} \geq 1.
\]

(\textit{Vasile Cîrtoaje, 2008})
Solution. Using the substitutions
\[ a = \sqrt[\frac{2}{yz}] x, \quad b = \sqrt[\frac{2}{3}] y, \quad c = \sqrt[\frac{2}{xy}] z, \]
the inequality becomes
\[ \sum \frac{x^4}{y^2z^2 + 2x^3 \sqrt[3]{xyz}} \geq 1. \]
By the Cauchy-Schwarz inequality, we have
\[ \sum \frac{x^4}{y^2z^2 + 2x^3 \sqrt[3]{xyz}} \geq \left( \sum x^2 \right)^2 \sum \frac{1}{y^2z^2 + 2x^3 \sqrt[3]{xyz}} = \frac{\left( \sum x^2 \right)^2}{\sum x^2y^2 + 2\sqrt[3]{xyz} \sum x^3}. \]
Therefore, we need to show that
\[ \frac{\left( \sum x^2 \right)^2}{\sum x^2y^2 + 2\sqrt[3]{xyz} \sum x^3}. \]
Since \( x + y + z \geq 3 \sqrt[3]{xyz} \), it suffices to prove that
\[ 3\left( \sum x^2 \right)^2 \geq 3 \sum x^2y^2 + 2 \left( \sum x \right) \left( \sum x^3 \right); \]
that is,
\[ \sum x^4 + 3 \sum x^2y^2 \geq 2 \sum xy(x^2 + y^2), \]
or, equivalently,
\[ \sum (x - y)^4 \geq 0. \]
The equality holds for \( a = b = c = 1 \).

P 1.48. Let \( a, b, c \) be positive real numbers such that \( abc = 1 \). Prove that
\[ \frac{a}{a^2 + 5} + \frac{b}{b^2 + 5} + \frac{c}{c^2 + 5} \leq \frac{1}{2}. \]

(Vasile Cîrtoaje, 2008)

Solution. Let
\[ F(a, b, c) = \frac{a}{a^2 + 5} + \frac{b}{b^2 + 5} + \frac{c}{c^2 + 5}. \]
Without loss of generality, assume that \( a = \min\{a, b, c\} \).

Case 1: \( a \leq 1/5 \). We have
\[ F(a, b, c) \leq \frac{a}{5} + \frac{b}{2\sqrt{5}b^2} + \frac{c}{2\sqrt{5}c^2} \leq \frac{1}{25} + \frac{1}{\sqrt{5}} < \frac{1}{2}. \]
Case 2: $a > 1/5$. Let $x = \sqrt{bc}$, $a = 1/x^2$, $x < \sqrt{5}$. We will show that

$$F(a, b, c) \leq F(a, x, x) \leq \frac{1}{2}.$$  

The left inequality, $F(a, b, c) \leq F(a, x, x)$, is equivalent to

$$(\sqrt{b} - \sqrt{c})^2[10x(b + c) + 10x^2 - 25 - x^4] \geq 0.$$  

This is true since

$$10x(b + c) + 10x^2 - 25 - x^4 \geq 20x^2 + 10x^2 - 25x^2 - x^4 = x^2(5 - x^2) > 0.$$  

The right inequality, $F(a, x, x) \leq 1/2$, is equivalent to

$$(x - 1)^2(5x^4 - 10x^3 - 2x^2 + 6x + 5) \geq 0.$$  

It is also true since

$$5x^4 - 10x^3 - 2x^2 + 6x + 5 = 5(x - 1)^4 + 2x(5x^2 - 16x + 13)$$  

and

$$5x^2 + 13 \geq 2\sqrt{65x^2} > 16x.$$  

The equality holds for $a = b = c = 1$.  

\[ \square \]

P 1.49. Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} + \frac{1}{(1 + c)^2} + \frac{2}{(1 + a)(1 + b)(1 + c)} \geq 1.$$  

(Pham Van Thuan, 2006)

**First Solution.** There are two of $a, b, c$ either greater than or equal to 1, or less than or equal to 1. Let $b$ and $c$ be these numbers; that is, $(1 - b)(1 - c) \geq 0$. Since

$$\frac{1}{(1 + b)^2} + \frac{1}{(1 + c)^2} \geq \frac{1}{1 + bc}$$  

(see the proof of P 1.45), it suffices to show that

$$\frac{1}{(1 + a)^2} + \frac{1}{1 + bc} + \frac{2}{(1 + a)(1 + b)(1 + c)} \geq 1.$$  

This inequality is equivalent to

$$\frac{b^2c^2}{(1 + bc)^2} + \frac{1}{1 + bc} + \frac{2bc}{(1 + bc)(1 + b)(1 + c)} \geq 1,$$
which can be written in the obvious form
\[
\frac{bc(1-b)(1-c)}{(1+bc)(1+b)(1+c)} \geq 0.
\]
The equality holds for \(a = b = c = 1\).

**Second Solution.** Setting \(a = \frac{yz}{x^2}, b = \frac{zx}{y^2}, c = \frac{xy}{z^2}\), where \(x, y, z > 0\), the inequality becomes
\[
\sum \frac{x^4}{(x^2 + yz)^2} + \frac{2x^2y^2z^2}{(x^2 + yz)(y^2 + zx)(z^2 + xy)} \geq 1.
\]
Since \((x^2 + yz)^2 \leq (x^2 + y^2)(x^2 + z^2)\), we have
\[
\sum \frac{x^4}{(x^2 + yz)^2} \geq \sum \frac{x^4}{(x^2 + y^2)(x^2 + z^2)} = 1 - \frac{2x^2y^2z^2}{(x^2 + y^2)(y^2 + zx)(z^2 + xy)}.
\]
Then, it suffices to show that
\[
(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \geq (x^2 + yz)(y^2 + zx)(z^2 + xy).
\]
This inequality follows by multiplying the inequalities
\[
(x^2 + y^2)(x^2 + z^2) \geq (x^2 + yz)^2,
\]
\[
(y^2 + z^2)(y^2 + x^2) \geq (y^2 + zx)^2,
\]
\[
(z^2 + x^2)(z^2 + y^2) \geq (z^2 + xy)^2.
\]

**Third Solution.** We make the substitutions
\[
\frac{1}{1+a} = \frac{1+x}{2}, \quad \frac{1}{1+b} = \frac{1+y}{2}, \quad \frac{1}{1+c} = \frac{1+z}{2},
\]
that is,
\[
a = \frac{1-x}{1+x}, \quad b = \frac{1-y}{1+y}, \quad c = \frac{1-z}{1+z},
\]
where \(-1 < x, y, z < 1\). Since \(abc = 1\) involves \(x + y + z + xyz = 0\), we need to prove that
\[
x + y + z + xyz = 0
\]
implies
\[
(1 + x)^2 + (1 + y)^2 + (1 + z)^2 + (1 + x)(1 + y)(1 + z) \geq 4.
\]
This inequality is equivalent to
\[
x^2 + y^2 + z^2 + (x + y + z)^2 + 4(x + y + z) \geq 0.
\]
By virtue of the AM-GM inequality, we have
\[ x^2 + y^2 + z^2 + (x + y + z)^2 + 4(x + y + z) = x^2 + y^2 + z^2 + x^2y^2z^2 - 4xyz \]
\[ \geq 4\sqrt[4]{x^4y^4z^4} - 4xyz = 4|xyz| - 4xyz \geq 0. \]
\[ \square \]

**P 1.50.** Let \( a, b, c \) be nonnegative real numbers such that
\[ \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} = \frac{3}{2}. \]
Prove that
\[ \frac{3}{a + b + c} \geq \frac{2}{ab + bc + ca} + \frac{1}{a^2 + b^2 + c^2}. \]

**Solution.** Write the inequality in the homogeneous form
\[ \frac{2}{a + b + c} \left( \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \right) \geq \frac{2}{ab + bc + ca} + \frac{1}{a^2 + b^2 + c^2}. \]
Denote \( q = ab + bc + ca \) and assume that \( a + b + c = 1 \). From the known inequality \( (a + b + c)^2 \geq 3(ab + bc + ca) \), we get \( 1 - 3q \geq 0 \). Rewrite the desired inequality as follows
\[ 2 \left( \frac{1}{1 - c} + \frac{1}{1 - a} + \frac{1}{1 - b} \right) \geq \frac{2}{q} + \frac{1}{1 - 2q}, \]
\[ \frac{2(q + 1)}{q - abc} \geq \frac{2 - 3q}{q(1 - 2q)}; \]
\[ q^2(1 - 4q) + (2 - 3q)abc \geq 0. \]
By Schur's inequality, we have
\[ (a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca), \]
\[ 1 - 4q \geq -9abc. \]
Then,
\[ q^2(1 - 4q) + (2 - 3q)abc \geq -9q^2abc + (2 - 3q)abc \]
\[ = (1 - 3q)(2 + 3q)abc \geq 0. \]
The equality holds for \( a = b = c = 1 \), and for \( a = 0 \) and \( b = c = 5/3 \) (or any cyclic permutation). \[ \square \]
**P 1.51.** Let $a, b, c$ be nonnegative real numbers such that

$$7(a^2 + b^2 + c^2) = 11(ab + bc + ca).$$

Prove that

$$\frac{51}{28} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2.$$

**Solution.** Due to homogeneity, we may assume that $b+c = 2$. Let us denote $x = bc$, $0 \leq x \leq 1$. By the hypothesis $7(a^2 + b^2 + c^2) = 11(ab + bc + ca)$, we get

$$x = \frac{7a^2 - 22a + 28}{25}.$$

Then, $x \leq 1$ involves $1/7 \leq a \leq 3$. Since

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a}{b+c} + \frac{a(b+c) + (b+c)^2 - 2bc}{a^2 + (b+c)a + bc} = \frac{a}{2} + \frac{2(a+2-x)}{a^2 + 2a + x} = \frac{4a^3 + 27a + 11}{8a^2 + 7a + 7},$$

the required inequalities become

$$\frac{51}{28} \leq \frac{4a^3 + 27a + 11}{8a^2 + 7a + 7} \leq 2.$$

We have

$$\frac{4a^3 + 27a + 11}{8a^2 + 7a + 7} - \frac{51}{28} = \frac{(7a-1)(4a-7)^2}{28(8a^2 + 7a + 7)} \geq 0$$

and

$$2 - \frac{4a^3 + 27a + 11}{8a^2 + 7a + 7} = \frac{(3-a)(2a-1)^2}{8a^2 + 7a + 7} \geq 0.$$ 

This completes the proof. The left inequality becomes an equality for $7a = b = c$ (or any cyclic permutation), while the right inequality is an equality for $a/3 = b = c$ (or any cyclic permutation). 

□

**P 1.52.** Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{10}{(a + b + c)^2}.$$
Solution. Assume that \( a = \min\{a, b, c\} \), and denote
\[
x = b + \frac{a}{2}, \quad y = c + \frac{a}{2},
\]
Since
\[
a^2 + b^2 \leq x^2, \quad b^2 + c^2 \leq x^2 + y^2, \quad c^2 + a^2 \leq y^2,
\]
\[
(a + b + c)^2 = (x + y)^2 \geq 4xy,
\]
it suffices to show that
\[
\frac{1}{x^2} + \frac{1}{x^2 + y^2} + \frac{1}{y^2} \geq \frac{5}{2xy}.
\]
We have
\[
\frac{1}{x^2} + \frac{1}{x^2 + y^2} + \frac{1}{y^2} = \frac{1}{x^2} + \frac{1}{y^2} - \frac{2}{xy} + \frac{1}{x^2 + y^2} - \frac{1}{2xy} = \frac{(x - y)^2}{x^2 y^2} - \frac{(x - y)^2}{2xy(x^2 + y^2)} = \frac{(x - y)^2(2x^2 - xy + 2y^2)}{2x^2 y^2(x^2 + y^2)} \geq 0.
\]
The equality holds for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[\square\]

**P 1.53.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \geq \frac{3}{\max\{ab, bc, ca\}}.
\]

Solution. Assume that \( a = \min\{a, b, c\} \), hence \( bc = \max\{ab, bc, ca\} \). Since
\[
\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \geq \frac{1}{b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2},
\]
it suffices to show that
\[
\frac{1}{b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2} \geq \frac{3}{bc}.
\]
We have
\[
\frac{1}{b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2} - \frac{3}{bc} = \frac{(b - c)^4}{b^2 c^2 (b^2 - bc + c^2)} \geq 0.
\]
The equality holds for \( a = b = c \), and also \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[\square\]
P 1.54. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(2a + b + c)}{b^2 + c^2} + \frac{b(2b + c + a)}{c^2 + a^2} + \frac{c(2c + a + b)}{a^2 + b^2} \geq 6.$$ 

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum a(2a + b + c) \geq \left(\sum a(2a + b + c)\right)^2 \sum a(2a + b + c)(b^2 + c^2).$$

Thus, we still need to show that

$$2\left(\sum a^2 + \sum ab\right)^2 \geq 3 \sum a(2a + b + c)(b^2 + c^2),$$

which is equivalent to

$$2 \sum a^4 + 2abc \sum a + \sum ab(a^2 + b^2) \geq 6 \sum a^2 b^2.$$ 

We can obtain this inequality by adding Schur’s inequality of degree four

$$\sum a^4 + abc \sum a \geq \sum ab(a^2 + b^2)$$

and

$$\sum ab(a^2 + b^2) \geq 2 \sum a^2 b^2,$$

multiplied by 2 and 3, respectively. The equality occurs for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

\[\square\]

P 1.55. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2(b + c)^2}{b^2 + c^2} + \frac{b^2(c + a)^2}{c^2 + a^2} + \frac{c^2(a + b)^2}{a^2 + b^2} \geq 2(ab + bc + ca).$$

Solution. We apply the SOS method. Since

$$\frac{a^2(b + c)^2}{b^2 + c^2} = a^2 + \frac{2a^2bc}{b^2 + c^2},$$

we can write the inequality as

$$2\left(\sum a^2 - \sum ab\right) - \sum a^2 \left(1 - \frac{2bc}{b^2 + c^2}\right) \geq 0,$$
\[
\sum (b - c)^2 - \sum \frac{a^2(b - c)^2}{b^2 + c^2} \geq 0, \\
\sum \left(1 - \frac{a^2}{b^2 + c^2}\right) (b - c)^2 \geq 0.
\]

Without loss of generality, assume that \(a \geq b \geq c\). Since \(1 - \frac{c^2}{a^2 + b^2} > 0\), it suffices to prove that
\[
\left(1 - \frac{a^2}{b^2 + c^2}\right) (b - c)^2 + \left(1 - \frac{b^2}{c^2 + a^2}\right) (a - c)^2 \geq 0,
\]
which is equivalent to
\[
\frac{(a^2 - b^2 + c^2)(a - c)^2}{a^2 + c^2} \geq \frac{(a^2 - b^2 - c^2)(b - c)^2}{b^2 + c^2}.
\]

This inequality follows by multiplying the inequalities
\[
a^2 - b^2 + c^2 \geq a^2 - b^2 - c^2, \quad \frac{(a - c)^2}{a^2 + c^2} \geq \frac{(b - c)^2}{b^2 + c^2}.
\]

The latter inequality is true since
\[
\frac{(a - c)^2}{a^2 + c^2} - \frac{(b - c)^2}{b^2 + c^2} = \frac{2bc}{b^2 + c^2} - \frac{2ac}{a^2 + c^2} = \frac{2c(a - b)(ab - c^2)}{(b^2 + c^2)(a^2 + c^2)} \geq 0.
\]

The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.56.** If \(a, b, c\) are real numbers such that \(abc > 0\), then
\[
3 \sum \frac{a}{b^2 - bc + c^2} + 5\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) \geq 8\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).
\]

*(Vasile Cirtoaje, 2011)*

**Solution.** In order to apply the SOS method, we multiply the inequality by \(abc\) and write it as follows:
\[
8(\sum a^2 - \sum bc) - 3 \sum a^2 \left(1 - \frac{bc}{b^2 - bc + c^2}\right) \geq 0, \\
4 \sum (b - c)^2 - 3 \sum \frac{a^2(b - c)^2}{b^2 - bc + c^2} \geq 0, \\
\sum \frac{(b - c)^2(4b^2 - 4bc + 4c^2 - 3a^2)}{b^2 - bc + c^2} \geq 0.
\]
Without loss of generality, assume that \( a \geq b \geq c \). Since
\[
4a^2 - 4ab + 4b^2 - 3c^2 = (2a - b)^2 + 3(b^2 - c^2) \geq 0,
\]
it suffices to prove that
\[
\frac{(a - c)^2(4a^2 - 4ac + 4c^2 - 3b^2)}{a^2 - ac + c^2} \geq \frac{(b - c)^2(3a^2 - 4b^2 + 4bc - 4c^2)}{b^2 - bc + c^2}.
\]
Notice that
\[
4a^2 - 4ac + 4c^2 - 3b^2 = (a - 2c)^2 + 3(a^2 - b^2) \geq 0.
\]
Thus, the desired inequality follows by multiplying the inequalities
\[
4a^2 - 4ac + 4c^2 - 3b^2 \geq 3a^2 - 4b^2 + 4bc - 4c^2
\]
and
\[
\frac{(a - c)^2}{a^2 - ac + c^2} \geq \frac{(b - c)^2}{b^2 - bc + c^2}.
\]
The first inequality is equivalent to
\[
(a - 2c)^2 + (b - 2c)^2 \geq 0.
\]
Also, we have
\[
\frac{(a - c)^2}{a^2 - ac + c^2} - \frac{(b - c)^2}{b^2 - bc + c^2} = \frac{bc}{b^2 - bc + c^2} - \frac{ac}{a^2 - ac + c^2} = \frac{bc - ac}{c(a - b)(ab - c^2)} \geq 0.
\]
The equality occurs for \( a = b = c \), and for \( 2a = b = c \) (or any cyclic permutation).

**P 1.57.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[
(a) \quad 2abc \left( \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \right) + a^2 + b^2 + c^2 \geq 2(ab + bc + ca);
\]

\[
(b) \quad \frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + a} \leq \frac{3(a^2 + b^2 + c^2)}{2(a + b + c)}.
\]
Symmetric Rational Inequalities

Solution. (a) First Solution. We have

\[ 2abc \sum \frac{1}{b + c} + \sum a^2 = \sum \frac{a(2bc + ab + ac)}{b + c} \]
\[ = \sum \frac{ab(a + c)}{b + c} + \sum \frac{ac(a + b)}{b + c} \]
\[ = \sum \frac{ab(a + c)}{b + c} + \sum \frac{ba(b + c)}{c + a} \]
\[ = \sum ab \left( \frac{a + c}{b + c} + \frac{b + c}{a + c} \right) \geq 2 \sum ab. \]

The equality occurs for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

Second Solution. Write the inequality as

\[ \sum \left( \frac{2abc}{b + c} + a^2 - ab - ac \right) \geq 0. \]

We have

\[ \sum \left( \frac{2abc}{b + c} + a^2 - ab - ac \right) = \sum \frac{ab(a - b) + ac(a - c)}{b + c} \]
\[ = \sum \frac{ab(a - b)}{b + c} + \sum \frac{ba(b - a)}{c + a} \]
\[ = \sum \frac{ab(a - b)^2}{(b + c)(c + a)} \geq 0. \]

(b) Since

\[ \sum \frac{a^2}{a + b} = \sum \left( a - \frac{ab}{a + b} \right) = a + b + c - \sum \frac{ab}{a + b}, \]
we can write the desired inequality as

\[ \sum \frac{ab}{a + b} + \frac{3(a^2 + b^2 + c^2)}{2(a + b + c)} \geq a + b + c. \]

Multiplying by 2(a + b + c), the inequality can be written as

\[ 2 \sum \left( 1 + \frac{a}{b + c} \right) bc + 3(a^2 + b^2 + c^2) \geq 2(a + b + c)^2, \]

or

\[ 2abc \sum \frac{1}{b + c} + a^2 + b^2 + c^2 \geq 2(ab + bc + ca), \]

which is just the inequality in (a). \( \Box \)
P 1.58. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[
\begin{align*}
(a) & \quad \frac{a^2 - bc}{b^2 + c^2} + \frac{b^2 - ca}{c^2 + a^2} + \frac{c^2 - ab}{a^2 + b^2} + \frac{3(ab + bc + ca)}{a^2 + b^2 + c^2} \geq 3; \\
(b) & \quad \frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} + \frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq \frac{5}{2}; \\
(c) & \quad \frac{a^2 + bc}{b^2 + c^2} + \frac{b^2 + ca}{c^2 + a^2} + \frac{c^2 + ab}{a^2 + b^2} \geq \frac{ab + bc + ca}{a^2 + b^2 + c^2} + 2.
\end{align*}
\]

Solution. (a) Write the inequality as follows:

\[
\sum \left( \frac{2a^2}{b^2 + c^2} - 1 \right) + \sum \left( 1 - \frac{2bc}{b^2 + c^2} \right) - 6 \left( 1 - \frac{ab + bc + ca}{a^2 + b^2 + c^2} \right) \geq 0,
\]

\[
\sum \frac{2a^2 - b^2 - c^2}{b^2 + c^2} + \sum \frac{(b - c)^2}{b^2 + c^2} - 3 \sum \frac{(b - c)^2}{a^2 + b^2 + c^2} \geq 0.
\]

Since

\[
\sum \frac{2a^2 - b^2 - c^2}{b^2 + c^2} = \sum \frac{a^2 - b^2}{b^2 + c^2} + \sum \frac{a^2 - c^2}{b^2 + c^2} = \sum \frac{a^2 - b^2}{b^2 + c^2} + \sum \frac{b^2 - a^2}{c^2 + a^2}
\]

\[
= \sum \frac{(a^2 - b^2)^2}{(b^2 + c^2)(c^2 + a^2)} = \sum \frac{(b^2 - c^2)^2}{(a^2 + b^2)(a^2 + c^2)},
\]

we can write the inequality as

\[
\sum (b - c)^2 S_a \geq 0,
\]

where

\[
S_a = \frac{(b + c)^2}{(a^2 + b^2)(a^2 + c^2)} + \frac{1}{b^2 + c^2} - \frac{3}{a^2 + b^2 + c^2}.
\]

It suffices to show that \( S_a \geq 0 \) for all nonnegative real numbers \( a, b, c \), no two of which are zero. Denoting \( x^2 = b^2 + c^2 \), we have

\[
S_a = \frac{x^2 + 2bc}{a^4 + a^2x^2 + b^2c^2} + \frac{1}{x^2} - \frac{3}{a^2 + x^2},
\]

and the inequality \( S_a \geq 0 \) becomes

\[
(a^2 - 2x^2)b^2c^2 + 2x^2(a^2 + x^2)bc + (a^2 + x^2)(a^2 - x^2)^2 \geq 0.
\]

Clearly, this is true if

\[-2x^2b^2c^2 + 2x^4bc \geq 0.\]

(Vasile Cîrtoaje, 2014)
Indeed,
\[-2x^2b^2c^2 + 2x^4bc = 2x^2bc(x^2 - bc) = 2bc(b^2 + c^2)(b^2 + c^2 - bc) \geq 0.\]
The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

(b) First Solution. We get the desired inequality by summing the inequality in (a) and the inequality
\[
\frac{bc}{b^2 + c^2} + \frac{ca}{c^2 + a^2} + \frac{ab}{a^2 + b^2} + \frac{1}{2} \geq \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2}.
\]
This inequality is equivalent to
\[
\sum \left( \frac{2bc}{b^2 + c^2} + 1 \right) \geq \frac{4(ab + bc + ca)}{a^2 + b^2 + c^2} + 2,
\]
and
\[
\sum \left( \frac{b + c}{b^2 + c^2} \right)^2 \geq \frac{2(a + b + c)^2}{a^2 + b^2 + c^2}.
\]
By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{(b + c)^2}{b^2 + c^2} \geq \left[ \sum (b + c) \right]^2 \sum (b^2 + c^2) = \frac{2(a + b + c)^2}{a^2 + b^2 + c^2}.
\]
The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

Second Solution. Let
\[
p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.
\]
By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{a^2}{b^2 + c^2} \geq \frac{\left( \sum a^2 \right)^2}{\sum a^2(b^2 + c^2)} = \frac{(p^2 - 2q)^2}{2(q^2 - 2pr)}.
\]
Therefore, it suffices to show that
\[
\frac{(p^2 - 2q)^2}{q^2 - 2pr} + \frac{2q}{p^2 - 2q} \geq 5. \tag{*}
\]
Consider the following cases: \(p^2 \geq 4q\) and \(3q \leq p^2 < 4q\).

Case 1: \(p^2 \geq 4q\). The inequality (*) is true if
\[
\frac{(p^2 - 2q)^2}{q^2} + \frac{2q}{p^2 - 2q} \geq 5,
\]
which is equivalent to the obvious inequality

\[(p^2 - 4q)[(p^2 - q)^2 - 2q^2] \geq 0.\]

Case 2: \(3q \leq p^2 < 4q\). Using Schur's inequality of degree four

\[6pq \geq (p^2 - q)(4q - p^2),\]

the inequality (*) is true if

\[
\frac{3(p^2 - 2q)^2}{3q^2 - (p^2 - q)(4q - p^2)} + \frac{2q}{p^2 - 2q} \geq 5,
\]

which is equivalent to the obvious inequality

\[(p^2 - 3q)(p^2 - 4q)(2p^2 - 5q) \leq 0.\]

**Third Solution** (by Nguyen Van Quy). Write the inequality (*) from the preceding solution as follows:

\[
\frac{(a^2 + b^2 + c^2)^2}{a^2b^2 + b^2c^2 + c^2a^2} + \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2} \geq 5,
\]

\[
\frac{(a^2 + b^2 + c^2)^2}{a^2b^2 + b^2c^2 + c^2a^2} - 3 \geq 2 - \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2},
\]

\[
\frac{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}{a^2b^2 + b^2c^2 + c^2a^2} \geq \frac{2(a^2 + b^2 + c^2 - ab - bc - ca)}{a^2 + b^2 + c^2}.
\]

Since

\[2(a^2b^2 + b^2c^2 + c^2a^2) \leq \sum ab(a^2 + b^2) \leq (ab + bc + ca)(a^2 + b^2 + c^2),\]

it suffices to prove that

\[
\frac{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}{ab + bc + ca} \geq a^2 + b^2 + c^2 - ab - bc - ca,
\]

which is just Schur's inequality of degree four

\[a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2).\]

(c) We get the desired inequality by summing the inequality in (a) and the inequality

\[
\frac{2bc}{b^2 + c^2} + \frac{2ca}{c^2 + a^2} + \frac{2ab}{a^2 + b^2} + 1 \geq \frac{4(ab + bc + ca)}{a^2 + b^2 + c^2},
\]

which is proved at (b). The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]
P 1.59. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \geq \frac{(a + b + c)^2}{2(ab + bc + ca)}.
\]

Solution. Applying the Cauchy-Schwarz inequality, we have

\[
\sum \frac{a^2}{b^2 + c^2} \geq \frac{(\sum a^2)^2}{\sum a^2(b^2 + c^2)} = \frac{(a^2 + b^2 + c^2)^2}{2(a^2b^2 + b^2c^2 + c^2a^2)}.
\]

Therefore, it suffices to show that

\[
\frac{(a^2 + b^2 + c^2)^2}{2(a^2b^2 + b^2c^2 + c^2a^2)} \geq \frac{(a + b + c)^2}{2(ab + bc + ca)},
\]

which is equivalent to

\[
\frac{(a^2 + b^2 + c^2)^2}{a^2b^2 + b^2c^2 + c^2a^2} - 3 \geq \frac{(a + b + c)^2}{ab + bc + ca} - 3,
\]

\[
\frac{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}{a^2b^2 + b^2c^2 + c^2a^2} \geq \frac{a^2b^2 + b^2c^2 - ab - bc - ca}{ab + bc + ca}.
\]

Since \( a^2b^2 + b^2c^2 + c^2a^2 \leq (ab + bc + ca)^2 \), it suffices to show that

\[
a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq (a^2 + b^2 + c^2 - ab - bc - ca)(ab + bc + ca),
\]

which is just Schur's inequality of degree four

\[
a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2).
\]

The equality holds for \( a = b = c \), and also for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[\square\]

P 1.60. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{2ab}{(a + b)^2} + \frac{2bc}{(b + c)^2} + \frac{2ca}{(c + a)^2} + \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq \frac{5}{2}.
\]

\[\text{(Vasile Cirtoaje, 2006)}\]
**First Solution.** We use the SOS method. Write the inequality as follows

\[ \frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \geq \sum \left( \frac{1}{2} - \frac{2bc}{(b + c)^2} \right), \]

\[ \sum \frac{(b-c)^2}{ab + bc + ca} \geq \sum \frac{(b-c)^2}{(b + c)^2}, \]

\[ (b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \geq 0, \]

where

\[ S_a = 1 - \frac{ab + bc + ca}{(b + c)^2}, \quad S_b = 1 - \frac{ab + bc + ca}{(c + a)^2}, \quad S_c = 1 - \frac{ab + bc + ca}{(a + b)^2}. \]

Without loss of generality, assume that \( a \geq b \geq c \). We have \( S_c > 0 \) and \( S_b \geq 1 - \frac{(c + a)(c + b)}{(c + a)^2} = \frac{a - b}{c + a} \geq 0 \).

If \( b^2 S_a + a^2 S_b \geq 0 \), then

\[ \sum (b-c)^2 S_a \geq (b-c)^2 S_a + (c-a)^2 S_b \geq (b-c)^2 S_a + \frac{a^2}{b^2} (b-c)^2 S_b \]

\[ = \frac{(b-c)^2 (b^2 S_a + a^2 S_b)}{b^2} \geq 0. \]

We have

\[ b^2 S_a + a^2 S_b = a^2 + b^2 - (ab + bc + ca) \left[ \left( \frac{b}{b + c} \right)^2 + \left( \frac{a}{c + a} \right)^2 \right] \]

\[ \geq a^2 + b^2 - (b + c)(c + a) \left[ \left( \frac{b}{b + c} \right)^2 + \left( \frac{a}{c + a} \right)^2 \right] \]

\[ = a^2 \left( 1 - \frac{b + c}{c + a} \right) + b^2 \left( 1 - \frac{c + a}{b + c} \right) \]

\[ = \frac{(a - b)^2 (ab + bc + ca)}{(b + c)(c + a)} \geq 0. \]

The equality occurs for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

**Second Solution.** Multiplying by \( ab + bc + ca \), the inequality becomes

\[ \sum \frac{2a^2 b^2}{(a + b)^2} + 2abc \sum \frac{1}{a + b} + a^2 + b^2 + c^2 \geq \frac{5}{2} (ab + bc + ca), \]

\[ 2abc \sum \frac{1}{a + b} + a^2 + b^2 + c^2 - 2(ab + bc + ca) - \sum \frac{1}{2} ab \left[ 1 - \sum \frac{4ab}{(a + b)^2} \right] \geq 0. \]
According to the second solution of P 1.57-(a), we can write the inequality as follows
\[
\sum \frac{ab(a-b)^2}{(b+c)(c+a)} - \sum \frac{ab(a-b)^2}{2(a+b)^2} \geq 0,
\]
\[(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \geq 0,
\]
where
\[S_a = \frac{bc}{b+c}[2(b+c)^2 - (a+b)(a+c)].\]

Without loss of generality, assume that \(a \geq b \geq c\). We have \(S_c > 0\) and
\[S_b = \frac{2ac^2(a+c)}{a+c} \geq 0.
\]
If \(S_a + S_b \geq 0\), then
\[
\sum (b-c)^2 S_a + (c-a)^2 S_b \geq (b-c)^2(S_a + S_b) \geq 0.
\]
The inequality \(S_a + S_b \geq 0\) is true if
\[
\frac{ac}{a+c}[2(a+c)^2 - (a+b)(b+c)] \geq \frac{bc}{b+c}[(a+b)(a+c) - 2(b+c)^2].
\]
Since
\[
\frac{ac}{a+c} \geq \frac{bc}{b+c},
\]

it suffices to show that
\[2(a+c)^2 - (a+b)(b+c) \geq (a+b)(a+c) - 2(b+c)^2.
\]
This is true since is equivalent to
\[(a-b)^2 + 2c(a+b) + 4c^2 \geq 0.
\]

\[\square\]

**P 1.61.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} + \frac{1}{4} \geq \frac{ab + bc + ca}{a^2 + b^2 + c^2}.
\]

\[(\text{Vasile Cîrtoaje, 2011)}\]
**First Solution.** We use the SOS method. Write the inequality as follows

\[
1 - \frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq \sum \left[ \frac{1}{4} - \frac{bc}{(b + c)^2} \right],
\]

\[
2 \sum \frac{(b - c)^2}{a^2 + b^2 + c^2} \geq \sum \frac{(b - c)^2}{(b + c)^2},
\]

\[
\sum (b - c)^2 \left[ 2 - \frac{a^2 + b^2 + c^2}{(b + c)^2} \right] \geq 0.
\]

Since

\[
2 - \frac{a^2 + b^2 + c^2}{(b + c)^2} = 1 + \frac{2bc - a^2}{(b + c)^2} \geq 1 - \left( \frac{a}{b + c} \right)^2,
\]

it suffices to show that

\[
(b - c)^2 S_a + (c - a)^2 S_b + (a - b)^2 S_c \geq 0,
\]

where

\[
S_a = 1 - \left( \frac{a}{b + c} \right)^2, \quad S_b = 1 - \left( \frac{b}{c + a} \right)^2, \quad S_c = 1 - \left( \frac{c}{a + b} \right)^2.
\]

Without loss of generality, assume that \(a \geq b \geq c\). Since \(S_b \geq 0\) and \(S_c > 0\), if \(b^2 S_a + a^2 S_b \geq 0\), then

\[
\sum (b - c)^2 S_a \geq (b - c)^2 S_a + (c - a)^2 S_b \geq (b - c)^2 S_a + \frac{a^2}{b^2} (b - c)^2 S_b
\]

\[
= \frac{(b - c)^2(b^2 S_a + a^2 S_b)}{b^2} \geq 0.
\]

We have

\[
b^2 S_a + a^2 S_b = a^2 + b^2 - \left( \frac{ab}{b + c} \right)^2 - \left( \frac{ab}{c + a} \right)^2
\]

\[
= a^2 \left[ 1 - \left( \frac{b}{b + c} \right)^2 \right] + b^2 \left[ 1 - \left( \frac{a}{c + a} \right)^2 \right] \geq 0.
\]

The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

**Second Solution.** Since \((a + b)^2 \leq 2(a^2 + b^2)\), it suffices to prove that

\[
\sum \frac{ab}{2(a^2 + b^2)} + \frac{1}{4} \geq \frac{ab + bc + ca}{a^2 + b^2 + c^2},
\]

which is equivalent to

\[
\sum \frac{2ab}{a^2 + b^2} + 1 \geq \frac{4(ab + bc + ca)}{a^2 + b^2 + c^2},
\]
\[
\sum \frac{(a + b)^2}{a^2 + b^2} \geq 2 + \frac{4(ab + bc + ca)}{a^2 + b^2 + c^2},
\]
\[
\sum \frac{(a + b)^2}{a^2 + b^2} \geq \frac{2(a + b + c)^2}{a^2 + b^2 + c^2}.
\]

The last inequality follows immediately by the Cauchy-Schwarz inequality
\[
\sum \frac{(a + b)^2}{a^2 + b^2} \geq \left(\sum (a + b)\right)^2 \sum (a^2 + b^2).
\]

**Remark.** The following generalization of the inequalities in P 1.60 and P 1.61 holds:
- Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(0 \leq k \leq 2\), then
\[
\sum \frac{4ab}{(a + b)^2} + k \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq 3k - 1 + 2(2 - k) \frac{ab + bc + ca}{a^2 + b^2 + c^2}.
\]

with equality for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.62.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{3ab}{(a + b)^2} + \frac{3bc}{(b + c)^2} + \frac{3ca}{(c + a)^2} \leq \frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{5}{4}.
\]

(Vasile Cîrtoaje, 2011)

**Solution.** We use the SOS method. Write the inequality as follows
\[
3 \sum \left[\frac{1}{4} - \frac{bc}{(b + c)^2}\right] \geq 1 - \frac{ab + bc + ca}{a^2 + b^2 + c^2},
\]
\[
3 \sum \frac{(b - c)^2}{(b + c)^2} \geq 2 \sum \frac{(b - c)^2}{a^2 + b^2 + c^2},
\]
\[
(b - c)^2 S_a + (c - a)^2 S_b + (a - b)^2 S_c \geq 0,
\]

where
\[
S_a = \frac{3(a^2 + b^2 + c^2)}{(b + c)^2} - 2, \quad S_b = \frac{3(a^2 + b^2 + c^2)}{(c + a)^2} - 2, \quad S_c = \frac{3(a^2 + b^2 + c^2)}{(a + b)^2} - 2.
\]

Without loss of generality, assume that \(a \geq b \geq c\). Since \(S_a > 0\) and
\[
S_b = \frac{a^2 + 3b^2 + c^2 - 4ac}{(c + a)^2} = \frac{(a - 2c)^2 + 3(b^2 - c^2)}{(c + a)^2} > 0,
\]
if \( S_b + S_c \geq 0 \), then

\[
\sum (b-c)^2 S_a \geq (c-a)^2 S_b + (a-b)^2 S_c \geq (a-b)^2 (S_b + S_c) \geq 0.
\]

Using the Cauchy-Schwarz Inequality, we have

\[
S_b + S_c = 3(a^2 + b^2 + c^2) \left[ \frac{1}{(c+a)^2} + \frac{1}{(a+b)^2} \right] - 4 \\
\geq \frac{12(a^2 + b^2 + c^2)}{(c+a)^2 + (a+b)^2} - 4 = \frac{4(a-b-c)^2 + 4(b-c)^2}{(c+a)^2 + (a+b)^2} \geq 0.
\]

The equality occurs for \( a = b = c \), and for \( \frac{a}{2} = b = c \) (or any cyclic permutation).

**P 1.63.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[
\begin{align*}
(a) \quad \frac{a^3 + abc}{b+c} + \frac{b^3 + abc}{c+a} + \frac{c^3 + abc}{a+b} & \geq a^2 + b^2 + c^2; \\
(b) \quad \frac{a^3 + 2abc}{b+c} + \frac{b^3 + 2abc}{c+a} + \frac{c^3 + 2abc}{a+b} & \geq \frac{1}{2}(a+b+c)^2; \\
(c) \quad \frac{a^3 + 3abc}{b+c} + \frac{b^3 + 3abc}{c+a} + \frac{c^3 + 3abc}{a+b} & \geq 2(ab + bc + ca).
\end{align*}
\]

**Solution.** (a) First Solution. Write the inequality as

\[
\sum \left( \frac{a^3 + abc}{b+c} - a^2 \right) \geq 0,
\]

\[
\sum \frac{a(a-b)(a-c)}{b+c} \geq 0.
\]

Assume that \( a \geq b \geq c \). Since \((c-a)(c-b) \geq 0\) and

\[
\frac{a(a-b)(a-c)}{b+c} + \frac{b(b-c)(b-a)}{b+c} = \frac{(a-b)^2(a^2 + b^2 + c^2 + ab)}{(b+c)(c+a)} \geq 0,
\]

the conclusion follows. The equality occurs for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

(b) Taking into account the inequality in (a), it suffices to show that

\[
\frac{abc}{b+c} + \frac{abc}{c+a} + \frac{abc}{a+b} + a^2 + b^2 + c^2 \geq \frac{1}{2}(a+b+c)^2,
\]
which is just the inequality (a) from P 1.57. The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

(c) The desired inequality follows by adding the inequality in (a) to the inequality (a) from P 1.57. The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.64.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{a^3 + 3abc}{(b + c)^2} + \frac{b^3 + 3abc}{(c + a)^2} + \frac{c^3 + 3abc}{(a + b)^2} \geq a + b + c.
\]

*(Vasile Cîrtoaje, 2005)*

**Solution.** We use the SOS method. We have

\[
\sum_a \frac{a^3 + 3abc}{(b + c)^2} - \sum_a = \sum_a \left[ \frac{a^3 + 3abc}{(b + c)^2} - a \right] = \sum \frac{a^3 - a(b^2 - bc + c^2)}{(b + c)^2}
\]

\[
= \sum \frac{a^3(b + c) - a(b^3 + c^3)}{(b + c)^3} = \sum \frac{ab(a^2 - b^2) + ac(a^2 - c^2)}{(b + c)^3}
\]

\[
= \sum \frac{ab(a^2 - b^2)}{(b + c)^3} + \sum \frac{ba(b^2 - a^2)}{(c + a)^3} = \sum \frac{ab(a^2 - b^2)((c + a)^3 - (b + c)^3)}{(b + c)^3(c + a)^3}
\]

\[
= \sum \frac{ab(a + b)(a - b)^2[(c + a)^2 + (c + a)(b + c) + (b + c)^2]}{(b + c)^2(c + a)^3} \geq 0.
\]

The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.65.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

(a) \[
\frac{a^3 + 3abc}{(b + c)^3} + \frac{b^3 + 3abc}{(c + a)^3} + \frac{c^3 + 3abc}{(a + b)^3} \geq \frac{3}{2};
\]

(b) \[
\frac{3a^2 + 13abc}{(b + c)^3} + \frac{3b^2 + 13abc}{(c + a)^3} + \frac{3c^2 + 13abc}{(a + b)^3} \geq 6.
\]

*(Vasile Cîrtoaje and Ji Chen, 2005)*
Solution. (a) First Solution. Use the SOS method. We have
\[
\sum \frac{a^3 + 3abc}{(b+c)^3} = \sum a(b+c)^2 + a(a^2 + bc - b^2 - c^2)
\]
\[
= \sum \frac{a}{b+c} + \sum \frac{a^3 - a(b^2 - bc + c^2)}{(b+c)^3}
\]
\[
\geq \frac{3}{2} + \sum \frac{a^3(b+c) - a(b^3 + c^3)}{(b+c)^4}
\]
\[
= \frac{3}{2} + \sum \frac{ab(a^2 - b^2) + ac(a^2 - c^2)}{(b+c)^4}
\]
\[
= \frac{3}{2} + \sum \frac{ab(a^2 - b^2)}{(b+c)^4} + \sum \frac{ba(b^2 - a^2)}{(c+a)^4}
\]
\[
= \frac{3}{2} + \sum \frac{ab(a+b)(a-b)(c+a)(c+b)}{(b+c)^4(c+a)^4} \geq 0.
\]
The equality occurs for \(a = b = c\).

Second Solution. Assume that \(a \geq b \geq c\). Since
\[
\frac{a^3 + 3abc}{b+c} \geq \frac{b^3 + 3abc}{c+ac} \geq \frac{c^3 + 3abc}{a+b}
\]
and
\[
\frac{1}{(b+c)^2} \geq \frac{1}{(c+a)^2} \geq \frac{1}{(a+b)^2},
\]
by Chebyshev’s inequality, we get
\[
\sum \frac{a^3 + 3abc}{(b+c)^3} \geq \frac{1}{3} \left( \sum \frac{a^3 + 3abc}{b+c} \right) \sum \frac{1}{(b+c)^2}.
\]
Thus, it suffices to show that
\[
\left( \sum \frac{a^3 + 3abc}{b+c} \right) \sum \frac{1}{(b+c)^2} \geq \frac{9}{2}.
\]
We can obtain this inequality by multiplying the known inequality (Iran-1996)
\[
\sum \frac{1}{(b+c)^2} \geq \frac{9}{4(ab+bc+ca)}
\]
and the inequality (c) from P 1.63.

(b) We have
\[
\sum \frac{3a^3 + 13abc}{(b+c)^3} = \sum \frac{3a(b+c)^2 + 4abc + 3a(a^2 + bc - b^2 - c^2)}{(b+c)^3}
\]
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\[ \sum \frac{3a}{b+c} + 4abc \sum \frac{1}{(b+c)^3} + 3 \sum \frac{a^3 - a(b^2 - bc + c^2)}{(b+c)^3}. \]

Since

\[ \sum \frac{1}{(b+c)^3} \geq \frac{3}{(a+b)(b+c)(c+a)} \]

and

\[ \sum \frac{a^3 - a(b^2 - bc + c^2)}{(b+c)^3} = \sum \frac{a^3(b+c) - a(b^3 + c^3)}{(b+c)^4} = \sum \frac{ab(a^2 - b^2) + ac(a^2 - c^2)}{(b+c)^4} = \sum \frac{ba(b^2 - a^2)}{(c+a)^4} \]

\[ = \sum \frac{ab(a+b)(a-b)[(c+a)^3 - (b+c)^4]}{b(c+a)^4} \geq 0, \]

it suffices to prove that

\[ \sum \frac{3a}{b+c} + \frac{12abc}{(a+b)(b+c)(c+a)} \geq 6. \]

This inequality is equivalent to the third degree Schur’s inequality

\[ a^3 + b^3 + c^3 + 3abc \geq \sum ab(a+b). \]

The equality occurs for \( a = b = c, \) and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[ \square \]

P 1.66. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

(a) \[ \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} + ab + bc + ca \geq \frac{3}{2}(a^2 + b^2 + c^2); \]

(b) \[ \frac{2a^2 + bc}{b+c} + \frac{2b^2 + ca}{c+a} + \frac{2c^2 + ab}{a+b} \geq \frac{9(a^2 + b^2 + c^2)}{2(a+b+c)}. \]

(Vasile Cîrtoaje, 2006)

Solution. (a) We apply the SOS method. Write the inequality as

\[ \sum \left( \frac{2a^3}{b+c} - a^2 \right) \geq \sum (a-b)^2. \]

Since

\[ \sum \left( \frac{2a^3}{b+c} - a^2 \right) = \sum \frac{a^2(a-b) + a^2(a-c)}{b+c} \]
\[ \sum \frac{a^2(a-b)}{b+c} + \sum \frac{b^2(b-a)}{c+a} = \sum \frac{(a-b)^2(a^2+b^2+ab+bc+ca)}{(b+c)(c+a)}, \]

we can write the inequality as

\[ (b-c)^2S_a + (c-a)^2S_b + (a-b)^2S_c \geq 0, \]

where

\[ S_a = (b+c)(b^2+c^2-a^2), \quad S_b = (c+a)(c^2+a^2-b^2), \quad S_c = (a+b)(a^2+b^2-c^2). \]

Without loss of generality, assume that \( a \geq b \geq c \). Since \( S_b \geq 0, S_c \geq 0, \) and \( S_a + S_b = (a+b)(a-b)^2 + c^2(a+b+2c) \geq 0, \)

we have

\[ \sum (b-c)^2S_a \geq (b-c)^2S_a + (a-c)^2S_b \geq (b-c)^2(S_a + S_b) \geq 0. \]

The equality occurs for \( a = b = c, \) and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

(b) Multiplying by \( a + b + c, \) the inequality can be written as

\[ \sum \left( 1 + \frac{a}{b+c} \right)(2a^2+bc) \geq \frac{9}{2}(a^2+b^2+c^2), \]

or

\[ \sum \frac{2a^3+abc}{b+c} + ab+bc+ca \geq \frac{5}{2}(a^2+b^2+c^2). \]

This inequality follows using the inequality in (a) and the first inequality from P 1.57.

The equality occurs for \( a = b = c, \) and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[ \square \]

**P 1.67.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[ \frac{a(b+c)}{b^2+bc+c^2} + \frac{b(c+a)}{c^2+ca+a^2} + \frac{c(a+b)}{a^2+ab+b^2} \geq 2. \]

**First Solution.** Apply the SOS method. We have

\[ (a+b+c) \left[ \sum \frac{a(b+c)}{b^2+bc+c^2} - 2 \right] = \sum \left[ \frac{a(b+c)(a+b+c)}{b^2+bc+c^2} - 2a \right] \]

\[ = \sum \frac{a(ab+ac-b^2-c^2)}{b^2+bc+c^2} = \sum \frac{ab(a-b)-ca(c-a)}{b^2+bc+c^2} \]
\[
\sum \frac{ab(a-b)}{b^2+bc+c^2} - \sum \frac{ab(a-b)}{c^2+ca+a^2}
= (a+b+c) \sum \frac{ab(a-b)^2}{(b^2+bc+c^2)(c^2+ca+a^2)} \geq 0.
\]

The equality occurs for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

**Second Solution.** By the AM-GM inequality, we have

\[
4(b^2+bc+c^2)(ab+bc+ca) \leq (b^2+bc+c^2 + ab+bc+ca)^2 = (b+c)^2(a+b+c)^2.
\]

Thus,

\[
\sum \frac{a(b+c)}{b^2+bc+c^2} = \sum \frac{a(b+c)(ab+bc+ca)}{(b^2+bc+c^2)(ab+bc+ca)}
\geq \sum \frac{4a(ab+bc+ca)}{(b+c)(a+b+c)^2} = \frac{4(ab+bc+ca)}{(a+b+c)^2} \sum \frac{a}{b+c},
\]

and it suffices to show that

\[
\sum \frac{a}{b+c} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}.
\]

This follows immediately from the Cauchy-Schwarz inequality

\[
\sum \frac{a}{b+c} \geq \frac{(a+b+c)^2}{\sum a(b+c)}.
\]

**Third Solution.** By the Cauchy-Schwarz inequality, we have

\[
\sum \frac{a(b+c)}{b^2+bc+c^2} \geq \frac{(a+b+c)^2}{\sum \frac{a(b^2+bc+c^2)}{b+c}},
\]

Thus, it is enough to show that

\[
(a+b+c)^2 \geq 2 \sum \frac{a(b^2+bc+c^2)}{b+c}.
\]

Since

\[
\frac{a(b^2+bc+c^2)}{b+c} = a\left(b+c - \frac{bc}{b+c}\right) = ab + ca - \frac{abc}{b+c},
\]

this inequality is equivalent to

\[
2abc \left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) + a^2 + b^2 + c^2 \geq 2(ab+bc+ca),
\]

which is just the inequality (a) from P 1.57.
Fourth Solution. By direct calculation, we can write the inequality as
\[ \sum ab(a^4 + b^4) \geq \sum a^2 b^2 (a^2 + b^2), \]
which is equivalent to the obvious inequality
\[ \sum ab(a - b)(a^3 - b^3) \geq 0. \]

\[ \blacksquare \]

P 1.68. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[ \frac{a(b + c)}{b^2 + bc + c^2} + \frac{b(c + a)}{c^2 + ca + a^2} + \frac{c(a + b)}{a^2 + ab + b^2} \geq 2 + 4 \prod \left(\frac{a - b}{a + b}\right)^2. \]

(Vasile Cirtoaje, 2011)

Solution. For \( b = c = 1 \), the inequality reduces to \( a(a - 1)^2 \geq 0 \). Assume further that \( a < b < c \). According to the first solution of P 1.67, we have
\[ \sum \frac{a(b + c)}{b^2 + bc + c^2} - 2 = \sum \frac{bc(b - c)^2}{(a^2 + ab + b^2)(a^2 + ac + c^2)}. \]
Therefore, it remains to show that
\[ \sum \frac{bc(b - c)^2}{(a^2 + ab + b^2)(a^2 + ac + c^2)} \geq 4 \prod \left(\frac{a - b}{a + b}\right)^2. \]
Since
\[ (a^2 + ab + b^2)(a^2 + ac + c^2) \leq (a + b)^2(a + c)^2, \]
it suffices to show that
\[ \sum \frac{bc(b - c)^2}{(a + b)^2(a + c)^2} \geq 4 \prod \frac{(a - b)^2}{(a + b)^2}, \]
which is equivalent to
\[ \sum \frac{bc(b + c)^2}{(a - b)^2(a - c)^2} \geq 4. \]
We have
\[ \sum \frac{bc(b + c)^2}{(a - b)^2(a - c)^2} \geq \frac{bc(b + c)^2}{(a - b)^2(a - c)^2} \geq \frac{bc(b + c)^2}{b^2 c^2} = \frac{(b + c)^2}{bc} \geq 4. \]
The equality occurs for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation). \[ \blacksquare \]
P 1.69. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{ab - bc + ca}{b^2 + c^2} + \frac{bc - ca + ab}{c^2 + a^2} + \frac{ca - ab + bc}{a^2 + b^2} \geq \frac{3}{2}.
\]

**Solution.** Use the SOS method. We have

\[
\sum \left( \frac{ab - bc + ca}{b^2 + c^2} - \frac{1}{2} \right) = \sum \frac{(b + c)(2a - b - c)}{2(b^2 + c^2)}
\]

\[
= \sum \frac{(b + c)(a - b)}{2(b^2 + c^2)} + \sum \frac{(b + c)(a - c)}{2(b^2 + c^2)}
\]

\[
= \sum \frac{(b + c)(a - b)}{2(b^2 + c^2)} + \sum \frac{(c + a)(b - a)}{2(c^2 + a^2)}
\]

\[
= \sum \frac{(a - b)^2(ab + bc + ca - c^2)}{2(b^2 + c^2)(c^2 + a^2)}.
\]

Since

\[
ab + bc + ca - c^2 = (b - c)(c - a) + 2ab \geq (b - c)(c - a),
\]

it suffices to show that

\[
\sum (a^2 + b^2)(a - b)^2(b - c)(c - a) \geq 0.
\]

This inequality is equivalent to

\[
(a - b)(b - c)(c - a) \sum (a - b)(a^2 + b^2) \geq 0,
\]

or

\[
(a - b)^2(b - c)^2(c - a)^2 \geq 0.
\]

The equality occurs for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation). \(\square\)

P 1.70. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(k > -2\), then

\[
\sum \frac{ab + (k-1)bc + ca}{b^2 + kbc + c^2} \geq \frac{3(k+1)}{k+2}.
\]

*(Vasile Cîrtoaje, 2005)*
First Solution. Apply the SOS method. Write the inequality as
\[
\sum ab + \left( k - 1 \right) bc + ca \geq \frac{k + 1}{k + 2},
\]
\[
\sum \frac{A}{b^2 + kbc + c^2} \geq 0,
\]
where
\[
A = (b + c)(2a - b - c) + k(ab + ac - b^2 - c^2).
\]
Since
\[
A = (b + c)[(a - b) + (a - c)] + k[b(a - b) + c(a - c)]
\]
\[
= (a - b)[(k + 1)b + c] + (a - c)[(k + 1)c + b],
\]
the inequality is equivalent to
\[
\sum \frac{(a - b)[(k + 1)b + c]}{b^2 + kbc + c^2} + \sum \frac{(a - c)[(k + 1)c + b]}{b^2 + kbc + c^2} \geq 0,
\]
\[
\sum \frac{(a - b)[(k + 1)b + c]}{b^2 + kbc + c^2} + \sum \frac{(b - a)[(k + 1)a + c]}{c^2 + kca + a^2} \geq 0,
\]
\[
\sum (b - c)^2 R_a S_a \geq 0,
\]
where
\[
R_a = b^2 + kbc + c^2, \quad S_a = a(b + c - a) + (k + 1)bc.
\]
Without loss of generality, assume that \( a \geq b \geq c \).

Case 1: \( k \geq -1 \). Since \( S_a \geq a(b + c - a) \), it suffices to show that
\[
\sum a(b + c - a)(b - c)^2 R_a \geq 0.
\]
We have
\[
\sum a(b + c - a)(b - c)^2 R_a \geq a(b + c - a)(b - c)^2 R_a + b(c + a - b)(c - a)^2 R_b
\]
\[
\geq (b - c)^2 [a(b + c - a)R_a + b(c + a - b)R_b].
\]
Thus, it is enough to prove that
\[
a(b + c - a)R_a + b(c + a - b)R_b \geq 0.
\]
Since \( b + c - a \geq -(c + a - b) \), we have
\[
a(b + c - a)R_a + b(c + a - b)R_b \geq (c + a - b)(bR_b - aR_a)
\]
\[
= (c + a - b)(a - b)(ab - c^2) \geq 0.
\]
Case 2: $-2 < k \leq 1$. Since
\[ S_a = (a - b)(c - a) + (k + 2)bc \geq (a - b)(c - a), \]
we have
\[ \sum (b - c)^2 R_a S_a \geq (a - b)(b - c)(c - a) \sum (b - c)R_a. \]
From
\[ \sum (b - c)R_a = \sum (b - c)[b^2 + bc + c^2 - (1 - k)bc] \]
\[ = \sum (b^3 - c^3) - (1 - k) \sum bc(b - c) \]
\[ = (1 - k)(a - b)(b - c)(c - a), \]
we get
\[ (a - b)(b - c)(c - a) \sum (b - c)R_a = (1 - k)(a - b)^2(b - c)^2(c - a)^2 \geq 0. \]
This completes the proof. The equality occurs for $a = b = c$, and for $a = 0$ and $b = c$
(or any cyclic permutation).

Second Solution. Let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$. Write the inequality in
the form $f_6(a, b, c) \geq 0$, where
\[ f_6(a, b, c) = (k + 2) \sum [a(b + c) + (k - 1)bc](a^2 + kab + b^2)(a^2 + kac + c^2) \]
\[ -3(k + 1) \prod (b^2 + kbc + c^2) \]
\[ = (k + 2) \sum [(k - 2)bc + q](kab - c^2 + p^2 - 2q)(kac - b^2 + p^2 - 2q) \]
\[ -3(k + 1) \prod (kbc - a^2 + p^2 - 2q). \]
Thus, $f_6(a, b, c)$ has the same highest coefficient $A$ as
\[ (k + 2)(k - 2)P_2(a, b, c) - 3(k + 1)P_3(a, b, c), \]
where
\[ P_2(a, b, c) = \sum bc(kab - c^2)(kac - b^2), \]
\[ P_3(a, b, c) = \prod (kbc - a^2). \]
According to Remark 2 from P 1.75 in Volume 1,
\[ A = (k + 2)(k - 2)P_2(1, 1, 1) - 3(k + 1)P_3(1, 1, 1) \]
\[ = 3(k + 2)(k - 2)(k - 1)^2 - 3(k + 1)(k - 1)^3 = -9(k - 1)^2. \]
Since $A \leq 0$, according to P 2.76-(a) in Volume 1, it suffices to prove the original inequality for $b = c = 1$, and for $a = 0$. For these cases, this inequality has the obvious forms

$$(k + 2)a(a - 1)^2 \geq 0$$

and

$$(b - c)^2[(k + 2)(b^2 + c^2) + (k^2 + k + 1)bc] \geq 0,$$

respectively.

**Remark.** For $k = 1$ and $k = 0$, from P 1.70, we get the inequalities in P 1.67 and P 1.69, respectively. Besides, for $k = 2$, we get the well-known inequality (Iran 1996):

$$\sum \frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \geq \frac{9}{4(ab + bc + ca)}.$$

\[\square\]

**P 1.71.** Let $a, b, c$ be nonnegative real numbers, no two of which are zero. If $k > -2$, then

$$\sum \frac{3bc - a(b + c)}{b^2 + kbc + c^2} \leq \frac{3}{k + 2}.$$

*(Vasile Cîrtoaje, 2011)*

**Solution.** Write the inequality in P 1.70 as

$$\sum \left[1 - \frac{ab + (k - 1)bc + ca}{b^2 + kbc + c^2}\right] \leq \frac{3}{k + 2},$$

$$\sum \frac{b^2 + c^2 + bc - a(b + c)}{b^2 + kbc + c^2} \leq \frac{3}{k + 2}.$$

Since $b^2 + c^2 \geq 2bc$, we get

$$\sum \frac{3bc - a(b + c)}{b^2 + kbc + c^2} \leq \frac{3}{k + 2},$$

which is just the desired inequality. The equality occurs for $a = b = c$.

\[\square\]

**P 1.72.** Let $a, b, c$ be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{ab + 1}{a^2 + b^2} + \frac{bc + 1}{b^2 + c^2} + \frac{ca + 1}{c^2 + a^2} \geq \frac{4}{3}.$$
Symmetric Rational Inequalities

**Solution.** Write the inequality in the homogeneous form $E(a, b, c) \geq 4$, where

$$E(a, b, c) = \frac{4ab + bc + ca}{a^2 + b^2} + \frac{4bc + ca + ab}{b^2 + c^2} + \frac{4ca + ab + bc}{c^2 + a^2}.$$

Without loss of generality, assume that $a = \min\{a, b, c\}$. We will show that

$$E(a, b, c) \geq E(0, b, c) \geq 4.$$

For $a = 0$, we have $E(a, b, c) = E(0, b, c)$, and for $a > 0$, we have

$$\frac{E(a, b, c) - E(0, b, c)}{a} = \frac{4b^2 + c(b - a)}{b(a^2 + b^2)} + \frac{b + c}{b^2 + c^2} + \frac{4c^2 + b(c - a)}{c(c^2 + a^2)} > 0.$$

Also,

$$E(0, b, c) - 4 = \frac{b}{c} + \frac{4bc}{b^2 + c^2} + \frac{c}{b} - 4 = \frac{(b - c)^4}{bc(b^2 + c^2)} \geq 0.$$

The equality holds for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

\[\square\]

**P 1.73.** Let $a, b, c$ be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{5ab + 1}{(a + b)^2} + \frac{5bc + 1}{(b + c)^2} + \frac{5ca + 1}{(c + a)^2} \geq 2.$$

**Solution.** Write the inequality as $E(a, b, c) \geq 6$, where

$$E(a, b, c) = \frac{16ab + bc + ca}{(a + b)^2} + \frac{16bc + ca + ab}{(b + c)^2} + \frac{16ca + ab + bc}{(c + a)^2}.$$

Without loss of generality, assume that $a \leq b \leq c$.

**Case 1:** $16b^2 \geq c(a + b)$. We will show that

$$E(a, b, c) \geq E(0, b, c) \geq 6.$$

For $a = 0$, we have $E(a, b, c) = E(0, b, c)$, and for $a > 0$, we have

$$\frac{E(a, b, c) - E(0, b, c)}{a} = \frac{16b^2 - c(a + b)}{b(a + b)^2} + \frac{1}{b + c} + \frac{16c^2 - b(a + c)}{c(c + a)^2} > 0.$$

Also,

$$E(0, b, c) - 6 = \frac{b}{c} + \frac{16bc}{(b + c)^2} + \frac{c}{b} - 6 = \frac{(b - c)^4}{bc(b + c)^2} \geq 0.$$
Case 2: \(16b^2 < c(a + b)\). We have
\[
E(a, b, c) - 6 > \frac{16ab + bc + ca}{(a + b)^2} - 6 > \frac{16ab + 16b^2}{(a + b)^2} - 6 = \frac{2(5b - 3a)}{a + b} > 0.
\]
The equality holds for \(a = 0\) and \(b = c = \sqrt{3}\) (or any cyclic permutation).

\(\square\)

**P 1.74.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{a^2 - bc}{2b^2 - 3bc + 2c^2} + \frac{b^2 - ca}{2c^2 - 3ca + 2a^2} + \frac{c^2 - ab}{2a^2 - 3ab + 2b^2} \geq 0.
\]

\(\text{(Vasile Cîrtoaje, 2005)}\)

**Solution.** The hint is applying the Cauchy-Schwarz inequality after we made the numerators of the fractions to be nonnegative and as small as possible. Thus, we write the inequality as
\[
\sum \left( \frac{a^2 - bc}{2b^2 - 3bc + 2c^2} + 1 \right) \geq 3,
\]
\[
\sum \frac{a^2 + 2(b - c)^2}{2b^2 - 3bc + 2c^2} \geq 3.
\]
Using the Cauchy-Schwarz inequality
\[
\sum \frac{a^2 + 2(b - c)^2}{2b^2 - 3bc + 2c^2} \geq \frac{(5 \sum a^2 - 4 \sum ab)^2}{\sum [a^2 + 2(b - c)^2](2b^2 - 3bc + 2c^2)},
\]
it suffices to prove that
\[
(5 \sum a^2 - 4 \sum ab)^2 \geq 3 \sum [a^2 + 2(b - c)^2](2b^2 - 3bc + 2c^2).
\]
This inequality is equivalent to
\[
\sum a^4 + abc \sum a + 2 \sum ab(a^2 + b^2) \geq 6 \sum a^2b^2.
\]
We can obtain it by summing Schur’s inequality of degree four
\[
\sum a^4 + abc \sum a \geq \sum ab(a^2 + b^2)
\]
to the obvious inequality
\[
3 \sum ab(a^2 + b^2) \geq 6 \sum a^2b^2.
\]
The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\(\square\)
P 1.75. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2 - bc}{b^2 - bc + c^2} + \frac{2b^2 - ca}{c^2 - ca + a^2} + \frac{2c^2 - ab}{a^2 - ab + b^2} \geq 3.$$ 

(Vasile Cîrtoaje, 2005)

Solution. Write the inequality such that the numerators of the fractions are nonnegative and as small as possible:

$$\sum \left( \frac{2a^2 - bc}{b^2 - bc + c^2} + 1 \right) \geq 6,$$

$$\sum \frac{2a^2 + (b-c)^2}{b^2 - bc + c^2} \geq 6.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{2a^2 + (b-c)^2}{b^2 - bc + c^2} \geq \frac{4(2\sum a^2 - \sum ab)^2}{\sum [2a^2 + (b-c)^2](b^2 - bc + c^2)}.$$ 

Thus, we still have to prove that

$$2(2\sum a^2 - \sum ab)^2 \geq 3 \sum [2a^2 + (b-c)^2](b^2 - bc + c^2).$$

This inequality is equivalent to

$$2 \sum a^4 + 2abc \sum a + \sum ab(a^2 + b^2) \geq 6 \sum a^2 b^2.$$ 

We can obtain it by summing up Schur’s inequality of degree four

$$\sum a^4 + abc \sum a \geq \sum ab(a^2 + b^2)$$

and

$$\sum ab(a^2 + b^2) \geq 2 \sum a^2 b^2,$$

multiplied by 2 and 3, respectively. The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

P 1.76. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{2b^2 - bc + 2c^2} + \frac{b^2}{2c^2 - ca + 2a^2} + \frac{c^2}{2a^2 - ab + 2b^2} \geq 1.$$ 

(Vasile Cîrtoaje, 2005)
**Solution.** By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{a^2}{2b^2 - bc + 2c^2} \geq \frac{(\sum a^2)^2}{\sum a^2(2b^2 - bc + 2c^2)}.
\]
Therefore, it suffices to show that
\[
(\sum a^2)^2 \geq \sum a^2(2b^2 - bc + 2c^2),
\]
which is equivalent to
\[
\sum a^4 + abc \sum a \geq 2 \sum a^2 b^2.
\]
This inequality follows by adding Schur's inequality of degree four
\[
\sum a^4 + abc \sum a \geq \sum ab(a^2 + b^2)
\]
and
\[
\sum ab(a^2 + b^2) \geq 2 \sum a^2 b^2.
\]
The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.77.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{4b^2 - bc + 2c^2} + \frac{1}{4c^2 - ca + 2a^2} + \frac{1}{4a^2 - ab + 2b^2} \geq \frac{9}{7(a^2 + b^2 + c^2)}.
\]
*(Vasile Cirtoaje, 2005)*

**Solution.** We use the SOS method. Without loss of generality, assume that \(a \geq b \geq c\). Write the inequality as
\[
\sum \left[ \frac{7(a^2 + b^2 + c^2)}{4b^2 - bc + 4c^2} - 3 \right] \geq 0,
\]
\[
\sum \frac{7a^2 - 5b^2 - 5c^2 + 3bc}{4b^2 - bc + 4c^2} \geq 0,
\]
\[
\sum \frac{5(2a^2 - b^2 - c^2) - 3(a^2 - bc)}{4b^2 - bc + 4c^2} \geq 0.
\]
Since
\[
2(a^2 - bc) = (a - b)(a + c) + (a - c)(a + b),
\]
we have
\[
10(2a^2 - b^2 - c^2) - 6(a^2 - bc) =
\]
\[
= 10(a^2 - b^2) - 3(a - b)(a + c) + 10(a^2 - c^2) - 3(a - c)(a + b)
\]
\[
= (a - b)(7a + 10b - 3c) + (a - c)(7a + 10c - 3b).
\]
Thus, we can write the desired inequality as follows

\[
\sum \frac{(a-b)(7a+10b-3c)}{4b^2-bc+4c^2} + \sum \frac{(a-c)(7a+10c-3b)}{4b^2-bc+4c^2} \geq 0,
\]

\[
\sum \frac{(a-b)(7a+10b-3c)}{4b^2-bc+4c^2} + \sum \frac{(b-a)(7b+10a-3c)}{4c^2-ca+4a^2} \geq 0,
\]

\[
\sum \frac{(a-b)^2(28a^2+28b^2-9c^2+68ab-19bc-19ca)}{(4b^2-bc+4c^2)(4c^2-ca+4a^2)},
\]

\[
(b-c)^2S_a + (c-a)^2S_b + (a-b)^2S_c \geq 0,
\]

where

\[S_a = (4b^2-bc+4c^2)[(b-a)(28b+9a) + c(-19a+68b+28c)],\]

\[S_b = (4c^2-ca+4a^2)[(a-b)(28a+9b) + c(-19b+68a+28c)],\]

\[S_c = (4a^2-ab+4b^2)[(c-b)(28c+9c) + a(68b-19c+28a)].\]

Since \(S_b \geq 0\) and \(S_c > 0\), it suffices to show that \(S_a + S_b \geq 0\). We have

\[S_a \geq (4b^2-bc+4c^2)[(b-a)(28b+9a)-19ac],\]

\[S_b \geq (4c^2-ca+4a^2)[(a-b)(28a+9b)+19ac],\]

\[19ac[(4c^2-ca+4a^2)-(4b^2-bc+4c^2)] = 19ac(a-b)(4a+4b-c) \geq 0,\]

and hence

\[S_a + S_b \geq (a-b)[-(4b^2-bc+4c^2)(28b+9a) + (4c^2-ca+4a^2)(28a+9b)]\]

\[= (a-b)^2[112(a^2+ab+b^2) + 76c^2-28c(a+b)] \geq 0.\]

The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation). \(\square\)

---

**P 1.78.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{2a^2+bc}{b^2+c^2} + \frac{2b^2+ca}{c^2+a^2} + \frac{2c^2+ab}{a^2+b^2} \geq \frac{9}{2}.
\]

(Vasile Cîrtoaje, 2005)
First Solution. We apply the SOS method. Since
\[
\sum \left[ \frac{2(2a^2 + bc)}{b^2 + c^2} - 3 \right] = 2 \sum \frac{2a^2 - b^2 - c^2}{b^2 + c^2} - \sum \frac{(b - c)^2}{b^2 + c^2}
\]
and
\[
\sum \frac{2a^2 - b^2 - c^2}{b^2 + c^2} = \sum \frac{a^2 - b^2}{b^2 + c^2} + \sum \frac{a^2 - c^2}{b^2 + c^2} = \sum \frac{a^2 - b^2}{b^2 + c^2} + \sum \frac{b^2 - a^2}{c^2 + a^2}
\]
\[
= \sum (a^2 - b^2) \left( \frac{1}{b^2 + c^2} - \frac{1}{c^2 + a^2} \right) = \sum \frac{(a^2 - b^2)^2}{(b^2 + c^2)(c^2 + a^2)}
\]
\[
\geq \sum \frac{(a - b)^2(a^2 + b^2)}{(b^2 + c^2)(c^2 + a^2)}
\]
we can write the inequality as
\[
2 \sum \frac{(b - c)^2(b^2 + c^2)}{(c^2 + a^2)(a^2 + b^2)} \geq \sum \frac{(b - c)^2}{b^2 + c^2},
\]
or
\[(b - c)^2S_a + (c - a)^2S_b + (c - a)^2S_c \geq 0,
\]
where
\[S_a = 2(b^2 + c^2)^2 - (c^2 + a^2)(a^2 + b^2).
\]
Assume that \(a \geq b \geq c\). We have
\[S_b = 2(c^2 + a^2)^2 - (a^2 + b^2)(b^2 + c^2)
\]
\[\geq 2(c^2 + a^2)(c^2 + b^2) - (a^2 + b^2)(b^2 + c^2)
\]
\[= (b^2 + c^2)(a^2 - b^2 + 2c^2) \geq 0,
\]
\[S_c = 2(a^2 + b^2)^2 - (b^2 + c^2)(c^2 + a^2) > 0,
\]
\[S_a + S_b = (a^2 - b^2)^2 + 2c^2(a^2 + b^2 + 2c^2) \geq 0.
\]
Therefore,
\[(b - c)^2S_a + (c - a)^2S_b + (c - a)^2S_c \geq (b - c)^2S_a + (c - a)^2S_b
\]
\[\geq (b - c)^2(S_a + S_b) \geq 0.
\]
The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

Second Solution. Since
\[bc \geq \frac{2b^2c^2}{b^2 + c^2},
\]
we have
\[
\sum \frac{2a^2 + bc}{b^2 + c^2} \geq \sum \frac{2a^2 + 2b^2c^2}{b^2 + c^2} = 2(a^2b^2 + b^2c^2 + c^2a^2) \sum \frac{1}{(b^2 + c^2)^2}.
\]

Therefore, it suffices to show that
\[
\sum \frac{1}{(b^2 + c^2)^2} \geq \frac{9}{4(a^2b^2 + b^2c^2 + c^2a^2)},
\]
which is just the known Iran-1996 inequality (see Remark from P 1.70).

**Third Solution.** We get the desired inequality by summing the inequality in P 1.58-(a), namely
\[
\frac{2a^2 - 2bc}{b^2 + c^2} + \frac{2b^2 - 2ca}{c^2 + a^2} + \frac{2c^2 - 2ab}{a^2 + b^2} + \frac{6(ab + bc + ca)}{a^2 + b^2 + c^2} \geq 6,
\]
and the inequality
\[
\frac{3bc}{b^2 + c^2} + \frac{3ca}{c^2 + a^2} + \frac{3ab}{a^2 + b^2} + \frac{3}{2} \geq \frac{6(ab + bc + ca)}{a^2 + b^2 + c^2}.
\]

This inequality is equivalent to
\[
\sum \left( \frac{2bc}{b^2 + c^2} + 1 \right) \geq \frac{4(ab + bc + ca)}{a^2 + b^2 + c^2} + 2,
\]
and
\[
\sum \frac{(b + c)^2}{b^2 + c^2} \geq \frac{2(a + b + c)^2}{a^2 + b^2 + c^2}.
\]

By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{(b + c)^2}{b^2 + c^2} \geq \left( \sum (b + c) \right)^2 \sum \frac{1}{b^2 + c^2} = \frac{2(a + b + c)^2}{a^2 + b^2 + c^2}.
\]

\[\square\]

**P 1.79.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{2a^2 + 3bc}{b^2 + bc + c^2} + \frac{2b^2 + 3ca}{c^2 + ca + a^2} + \frac{2c^2 + 3ab}{a^2 + ab + b^2} \geq 5.
\]

*(Vasile Cîrtoaje, 2005)*
Solution. We apply the SOS method. Write the inequality as

\[
\sum \left[ \frac{3(2a^2 + 3bc)}{b^2 + bc + c^2} - 5 \right] \geq 0,
\]

or

\[
\sum \frac{6a^2 + 4bc - 5b^2 - 5c^2}{b^2 + bc + c^2} \geq 0.
\]

Since

\[
2(a^2 - bc) = (a - b)(a + c) + (a - c)(a + b),
\]

we have

\[
6a^2 + 4bc - 5b^2 - 5c^2 = 5(2a^2 - b^2 - c^2) - 4(a^2 - bc)
\]

\[
= 5(a^2 - b^2) - 2(a - b)(a + c) + 5(a^2 - c^2) - 2(a - c)(a + b)
\]

\[
= (a - b)(3a + 5b - 2c) + (a - c)(3a + 5c - 2b).
\]

Thus, we can write the desired inequality as follows

\[
\sum \frac{(a - b)(3a + 5b - 2c)}{b^2 + bc + c^2} + \sum \frac{(a - c)(3a + 5c - 2b)}{b^2 + bc + c^2} \geq 0,
\]

\[
\sum \frac{(a - b)(3a + 5b - 2c)}{b^2 + bc + c^2} + \sum \frac{(b - a)(3b + 5a - 2c)}{c^2 + ca + a^2} \geq 0,
\]

\[
\sum \frac{(a - b)^2(3a^2 + 3b^2 - 4c^2 + 8ab + bc + ca)}{(b^2 + bc + c^2)(c^2 + ca + a^2)}
\]

\[
(b - c)^2S_a + (c - a)^2S_b + (a - b)^2S_c \geq 0,
\]

where

\[
S_a = (b^2 + bc + c^2)(3b^2 + 3c^2 - 4a^2 + ab + 8bc + ca).
\]

Assume that \(a \geq b \geq c\). Since

\[
S_b = (c^2 + ca + a^2)[(a - b)(3a + 4b) + c(8a + b + c)] \geq 0
\]

and

\[
S_c = (a^2 + ab + b^2)[(b - c)(3b + 4c) + a(8b + c + a)] > 0,
\]

it suffices to show that \(S_a + S_b \geq 0\). We have

\[
S_a + S_b \geq (b^2 + bc + c^2)(b - a)(3b + 4a) + (c^2 + ca + a^2)(a - b)(3a + 4b)
\]

\[
= (a - b)^2[3(a + b)(a + b + c) + ab - c^2] \geq 0.
\]

The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation). \(\square\)
P 1.80. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2 + 5bc}{(b + c)^2} + \frac{2b^2 + 5ca}{(c + a)^2} + \frac{2c^2 + 5ab}{(a + b)^2} \geq \frac{21}{4}. \quad (Vasile Cîrtoaje, 2005)$$

**Solution.** Use the SOS method. Write the inequality as follows

$$\sum \left[ \frac{2a^2 + 5bc}{(b + c)^2} - \frac{7}{4} \right] \geq 0,$$

$$\sum \frac{4(a^2 - b^2) + 4(a^2 - c^2) - 3(b - c)^2}{(b + c)^2} \geq 0,$$

$$4 \sum \frac{b^2 - c^2}{(c + a)^2} + 4 \sum \frac{c^2 - b^2}{(a + b)^2} - 3 \sum \frac{(b - c)^2}{(b + c)^2} \geq 0,$$

$$4 \sum \frac{(b - c)^2(b + c)(2a + b + c)}{(c + a)(a + b)^2} - 3 \sum \frac{(b - c)^2}{(b + c)^2} \geq 0.$$

Substituting $b + c = x$, $c + a = y$ and $a + b = z$, we can rewrite the inequality in the form

$$(y - z)^2S_x + (z - x)^2S_y + (x - y)^2S_z \geq 0,$$

where

$$S_x = 4x^3(y + z) - 3y^2z^2, \quad S_y = 4y^3(z + x) - 3z^2x^2, \quad S_z = 4z^3(x + y) - 3x^2y^2.$$

Without loss of generality, assume that $0 < x \leq y \leq z, z \leq x + y$. We have $S_z > 0$ and

$$S_y \geq 4x^2y(z + x) - 3x^2z(x + y) = x^2[4xy + z(y - 3x)] \geq 0,$$

since for the nontrivial case $y - 3x < 0$, we get

$$4xy + z(y - 3x) \geq 4xy + (x + y)(y - 3x) = x^2(3x + y)(y - x) \geq 0.$$

Thus, it suffices to show that $S_x + S_y \geq 0$. Since

$$S_x + S_y = 4xy(x^2 + y^2) + 4(x^3 + y^3)z - 3(x^2 + y^2)z^2 \geq 4xy(x^2 + y^2) + 4(x^3 + y^3)z - 3(x^2 + y^2)(x + y)z \geq 2xy(x + y)^2 + (x^2 - 4xy + y^2)(x + y)^2 = (x - y)^2(x + y)^2.$$

The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). □
\[ \sum \frac{2a^2 + (2k + 1)bc}{b^2 + kbc + c^2} \geq \frac{3(2k + 3)}{k + 2}. \]

(Vasile Cîrtoaje, 2005)

**First Solution.** There are two cases to consider.

**Case 1:** \(-2 < k \leq -1/2\). Write the inequality as

\[ \sum \left[ \frac{2a^2 + (2k + 1)bc}{b^2 + kbc + c^2} - \frac{2k + 1}{k + 2} \right] \geq \frac{6}{k + 2}, \]

\[ \sum \frac{2(k + 2)a^2 - (2k + 1)(b - c)^2}{b^2 + kbc + c^2} \geq 6. \]

Since \(2(k + 2)a^2 - (2k + 1)(b - c)^2 \geq 0\) for \(-2 < k \leq -1/2\), we can apply the Cauchy-Schwarz inequality. Thus, it suffices to show that

\[ \frac{[2(k + 2) \sum a^2 - (2k + 1) \sum (b - c)^2]^2}{\sum [2(k + 2)a^2 - (2k + 1)(b - c)^2](b^2 + kbc + c^2)} \geq 6, \]

which is equivalent to each of the following inequalities

\[ \sum [(1 - k) \sum a^2 + (2k + 1) \sum ab] \geq 3, \]

\[ 2(k + 2) \sum a^4 + 2(k + 2)abc \sum a - (2k + 1) \sum ab(a^2 + b^2) \geq 6 \sum a^2b^2, \]

\[ 2(k + 2)[\sum a^4 + abc \sum a - \sum ab(a^2 + b^2)] + 3 \sum ab(a - b)^2 \geq 0. \]

The last inequality is true since, by Schur’s inequality of degree four, we have

\[ \sum a^4 + abc \sum a - \sum ab(a^2 + b^2) \geq 0. \]

**Case 2:** \(k \geq -9/5\). Use the SOS method. Without loss of generality, assume that \(a \geq b \geq c\). Write the inequality as

\[ \sum \left[ \frac{2a^2 + (2k + 1)bc}{b^2 + kbc + c^2} - \frac{2k + 3}{k + 2} \right] \geq 0, \]

\[ \sum \frac{2(k + 2)a^2 - (2k + 1)(b^2 + c^2) + 2(k + 1)bc}{b^2 + kbc + c^2} \geq 0, \]

\[ \sum \frac{(2k + 3)(2a^2 - b^2 - c^2) - 2(k + 1)(a^2 - bc)}{b^2 + kbc + c^2} \geq 0. \]
Since
\[2(a^2 - bc) = (a - b)(a + c) + (a - c)(a + b),\]
we have
\[
(2k + 3)(2a^2 - b^2 - c^2) - 2(k + 1)(a^2 - bc) =
\]
\[
= (2k + 3)(a^2 - b^2) - (k + 1)(a - b)(a + c) + (2k + 3)(a^2 - c^2) - (k + 1)(a - c)(a + b)
\]
\[
= (a - b)[(k + 2)a + (2k + 3)b - (k + 1)c] + (a - c)[(k + 2)a + (2k + 3)c - (k + 1)b].
\]
Thus, we can write the desired inequality as
\[
\sum \frac{(a - b)[(k + 2)a + (2k + 3)b - (k + 1)c]}{b^2 + kbc + c^2} + \sum \frac{(a - c)[(k + 2)a + (2k + 3)c - (k + 1)b]}{b^2 + kbc + c^2} \geq 0,
\]
or
\[
\sum \frac{(a - b)[(k + 2)a + (2k + 3)b - (k + 1)c]}{b^2 + kbc + c^2} + \sum \frac{(b - a)[(k + 2)b + (2k + 3)a - (k + 1)c]}{c^2 + kca + a^2} \geq 0,
\]
or
\[
(b - c)^2R_aS_a + (c - a)^2R_bS_b + (a - b)^2R_cS_c \geq 0,
\]
where
\[
R_a = b^2 + kbc + c^2, \ R_b = c^2 + kca + a^2, \ R_c = a^2 + kab + b^2,
\]
\[
S_a = (k + 2)(b^2 + c^2) - (k + 1)a^2 + (3k + 5)bc + (k^2 + k - 1)a(b + c)
\]
\[
= -(a - b)[(k + 1)^2a + (k + 2)b] + c[(k^2 + k - 1)a + (3k + 5)b + (k + 2)c],
\]
\[
S_b = (k + 2)(c^2 + a^2) - (k + 1)b^2 + (3k + 5)ca + (k^2 + k - 1)b(c + a)
\]
\[
= (a - b)[(k + 2)a + (k + 1)^2b] + c[(3k + 5)a + (k^2 + k - 1)b + (k + 2)c],
\]
\[
S_c = (k + 2)(a^2 + b^2) - (k + 1)c^2 + (3k + 5)ab + (k^2 + k - 1)c(a + b)
\]
\[
= (k + 2)(a^2 + b^2) + (3k + 5)ab + c[(k^2 + k - 1)(a + b) - (k + 1)^2c]
\]
\[
\geq (5k + 9)ab + c[(k^2 + k - 1)(a + b) - (k + 1)^2c].
\]
We have \(S_b \geq 0\), since for the nontrivial case
\[
(3k + 5)a + (k^2 + k - 1)b + (k + 2)c < 0,
\]
we get
\[
S_b \geq (a - b)[(k + 2)a + (k + 1)^2b] + b[(3k + 5)a + (k^2 + k - 1)b + (k + 2)c]
\]
\[
= (k + 2)(a^2 - b^2) + (k + 2)^2ab + (k + 2)b^2 > 0.
\]
Also, we have \( S_c \geq 0 \) for \( k \geq -9/5 \), since
\[
(5k + 9)ab + c[(k^2 + k - 1)(a + b) - (k + 1)^2 c] \geq \\
\geq (5k + 9)ac + c[(k^2 + k - 1)(a + b) - (k + 1)^2 c] \\
= (k + 2)(k + 4)ac + (k^2 + k - 1)bc - (k + 1)^2 c^2 \\
\geq (2k^2 + 7k + 7)bc - (k + 1)^2 c^2 \\
\geq (k + 2)(k + 3)c^2 \geq 0.
\]
Therefore, it suffices to show that \( R_aS_a + R_bS_b \geq 0 \). Since
\[
bR_b - aR_a = (a - b)(ab - c^2) \geq 0
\]
and
\[
R_aS_a + R_bS_b \geq R_a(S_a + \frac{a}{b}S_b),
\]
it suffices to show that
\[
S_a + \frac{a}{b}S_b \geq 0.
\]
We have
\[
bS_a + aS_b = (k + 2)(a + b)(a - b)^2 + c f(a, b, c) \\
\geq 2(k + 2)b(a - b)^2 + c f(a, b, c),
\]
\[
S_a + \frac{a}{b}S_b \geq 2(k + 2)(a - b)^2 + \frac{c}{b}f(a, b, c),
\]
where
\[
f(a, b, c) = b[(k^2 + k - 1)a + (3k + 5)b] + a[(3k + 5)a + (k^2 + k - 1)b] \\
+(k + 2)c(a + b) = (3k + 5)(a^2 + b^2) + 2(k^2 + k - 1)ab + (k + 2)c(a + b).
\]
For the nontrivial case \( f(a, b, c) < 0 \), we have
\[
S_a + \frac{a}{b}S_b \geq 2(k + 2)(a - b)^2 + f(a, b, c) \\
\geq 2(k + 2)(a - b)^2 + (3k + 5)(a^2 + b^2) + 2(k^2 + k - 1)ab \\
= (5k + 9)(a^2 + b^2) + 2(k^2 - k - 5)ab \geq 2(k + 2)^2ab \geq 0.
\]
The proof is completed. The equality holds for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

**Second Solution.** Let
\[
p = a + b + c, \quad q = ab + bc + ca.
\]
Write the inequality as \( f_6(a, b, c) \geq 0 \), where

\[
f_6(a, b, c) = (k + 2) \sum [2a^2 + (2k + 1)bc](a^2 + kab + b^2)(a^2 + kac + c^2) \]

\[
-3(2k + 3) \prod (b^2 + kbc + c^2).
\]

Since

\[
(a^2 + kab + b^2)(a^2 + kac + c^2) = (p^2 - 2q + kab - c^2)(p^2 - 2q + kac - b^2),
\]

\[
b^2 + kbc + c^2 = p^2 - 2q + kbc - a^2,
\]

\( f_6(a, b, c) \) has the same highest coefficient \( A \) as

\[
(k + 2)P_2(a, b, c) - 3(2k + 3)P_3(a, b, c),
\]

where

\[
P_2(a, b, c) = \sum [2a^2 + (2k + 1)bc](kab - c^2)(kac - b^2), \quad P_3(a, b, c) = \prod (b^2 + kbc + c^2).
\]

According to Remark 2 from the proof of P 1.75 in Volume 1, we have

\[
A = (k + 2)P_2(1, 1, 1) - 3(2k + 3)P_3(1, 1, 1) = 9(2k + 3)(k - 1)^2.
\]

On the other hand,

\[
f_6(a, 1, 1) = 2(k + 2)a(a^2 + ka + 1)(a - 1)^2(a + k + 2) \geq 0,
\]

\[
f_6(0, b, c) = (b - c)^2 \left[ 2(k + 2)(b^2 + c^2)^2 + 2(k + 2)^2bc(b^2 + c^2) + (4k^2 + 6k - 1)b^2c^2 \right].
\]

For \(-2 < k \leq -3/2\), we have \( A \leq 0 \). According to P 2.76-(a) in Volume 1, it suffices to show that \( f_6(a, 1, 1) \geq 0 \) and \( f_6(0, b, c) \geq 0 \) for all \( a, b, c \geq 0 \). The first condition is clearly satisfied. The second condition is also satisfied since

\[
2(k + 2)(b^2 + c^2)^2 + (4k^2 + 6k - 1)b^2c^2 \geq [8(k + 2) + 4k^2 + 6k - 1]b^2c^2
\]

\[
= (4k^2 + 14k + 15)b^2c^2 \geq 0.
\]

For \( k > -3/2 \), when \( A \geq 0 \), we will apply the highest coefficient cancellation method. Consider two cases: \( p^2 \leq 4q \) and \( p^2 > 4q \).

**Case 1: \( p^2 \leq 4q \).** Since

\[
f_6(1, 1, 1) = f_6(0, 1, 1) = 0,
\]

define the homogeneous function

\[
P(a, b, c) = abc + B(a + b + c)^3 + C(a + b + c)(ab + bc + ca)
\]
such that \( P(1, 1, 1) = P(0, 1, 1) = 0 \); that is,

\[
P(a, b, c) = abc + \frac{1}{9}(a + b + c)^3 - \frac{4}{9}(a + b + c)(ab + bc + ca).
\]

We will prove the sharper inequality \( g_6(a, b, c) \geq 0 \), where

\[
g_6(a, b, c) = f_6(a, b, c) - 9(2k + 3)(k - 1)^2p^2(a, b, c).
\]

Clearly, \( g_6(a, b, c) \) has the highest coefficient \( A = 0 \). Then, according to Remark 1 from \( P 2.76 \) in Volume 1, it suffices to prove that \( g_6(a, 1, 1) \geq 0 \) for \( 0 \leq a \leq 4 \). We have

\[
P(a, 1, 1) = \frac{a(a - 1)^2}{9},
\]

hence

\[
g_6(a, 1, 1) = f_6(a, 1, 1) - 9(2k + 3)(k - 1)^2p^2(a, 1, 1) = \frac{a(a - 1)^2g(a)}{9},
\]

where

\[
g(a) = 18(k + 2)(a^2 + ka + 1)(a + k + 2) - (2k + 3)(k - 1)^2a(a - 1)^2.
\]

Since \( a^2 + ka + 1 \geq (k + 2)a \), it suffices to show that

\[
18(k + 2)^2(a + k + 2) \geq (2k + 3)(k - 1)^2(a - 1)^2.
\]

Also, since \( (a - 1)^2 \leq 2a + 1 \), it is enough to prove that \( h(a) \geq 0 \), where

\[
h(a) = 18(k + 2)^2(a + k + 2) - (2k + 3)(k - 1)^2(2a + 1).
\]

Since \( h(a) \) is a linear function, the inequality \( h(a) \geq 0 \) is true if \( h(0) \geq 0 \) and \( h(4) \geq 0 \). Setting \( x = 2k + 3, \ x > 0 \), we get

\[
h(0) = 18(k + 2)^3 - (2k + 3)(k - 1)^2 = \frac{1}{4}(8x^3 + 37x^2 + 2x + 9) > 0.
\]

Also,

\[
\frac{1}{9}h(4) = 2(k + 2)^2(k + 6) - (2k + 3)(k - 1)^2 = 3(7k^2 + 20k + 15) > 0.
\]

**Case 2:** \( p^2 > 4q \). We will prove the sharper inequality \( g_6(a, b, c) \geq 0 \), where

\[
g_6(a, b, c) = f_6(a, b, c) - 9(2k + 3)(k - 1)^2a^2b^2c^2.
\]
We see that \( g_6(a, b, c) \) has the highest coefficient \( A = 0 \). According to Remark 1 from P 2.76 in Volume 1, it suffices to prove that \( g_6(a, 1, 1) \geq 0 \) for \( a > 4 \) and \( g_6(0, b, c) \geq 0 \) for all \( b, c \geq 0 \). We have

\[
g_6(a, 1, 1) = f_6(a, 1, 1) - 9(2k + 3)(k - 1)^2a^2
\]

\[
= a[2(k + 2)(a^2 + ka + 1)(a - 1)^2(a + k + 2) - 9(2k + 3)(k - 1)^2a].
\]

Since

\[
a^2 + ka + 1 > (k + 2)a, \quad (a - 1)^2 > 9,
\]

it suffices to show that

\[
2(k + 2)^2(a + k + 2) \geq (2k + 3)(k - 1)^2.
\]

Indeed,

\[
2(k + 2)^2(a + k + 2) - (2k + 3)(k - 1)^2 > 2(k + 2)^2(k + 6) - (2k + 3)(k - 1)^2
\]

\[
= 3(7k^2 + 20k + 15) > 0.
\]

Also,

\[
g_6(0, b, c) = f_6(0, b, c) \geq 0.
\]

\[\square\]

**P 1.82.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If \( k > -2 \), then

\[
\sum \frac{3bc - 2a^2}{b^2 + kbc + c^2} \leq \frac{3}{k + 2}.
\]

*(Vasile Cîrtoaje, 2011)*

**First Solution.** Write the inequality as

\[
\sum \left[ \frac{2a^2 - 3bc}{b^2 + kbc + c^2} + \frac{3}{k + 2} \right] \geq \frac{6}{k + 2}
\]

\[
\sum \frac{2(k + 2)a^2 + 3(b - c)^2}{b^2 + kbc + c^2} \geq 6.
\]

Applying the Cauchy-Schwarz inequality, it suffices to show that

\[
\frac{[2(k + 2) \sum a^2 + 3 \sum (b - c)^2]^2}{\sum[2(k + 2)a^2 + 3(b - c)^2](b^2 + kbc + c^2)} \geq 6,
\]
which is equivalent to each of the following inequalities

\[
\frac{2((k+5) \sum a^2 - 3 \sum ab)^2}{\sum [2(k+2)a^2 + 3(b-c)^2](b^2 + kbc + c^2)} \geq 3,
\]

\[
2(k+8) \sum a^4 + 2(2k+19) \sum a^2 b^2 \geq 6(k+2)abc \sum a + 21 \sum ab(a^2 + b^2),
\]

\[
2(k+2)f(a, b, c) + 3g(a, b, c) \geq 0,
\]

where

\[
f(a, b, c) = \sum a^4 + 2 \sum a^2 b^2 - 3abc \sum a,
\]

\[
g(a, b, c) = 4 \sum a^4 + 10 \sum a^2 b^2 - 7 \sum ab(a^2 + b^2).
\]

We need to show that \( f(a, b, c) \geq 0 \) and \( g(a, b, c) \geq 0 \). Indeed,

\[
f(a, b, c) = (\sum a^2)^2 - 3abc \sum a \geq (\sum ab)^2 - 3abc \sum a \geq 0
\]

and

\[
g(a, b, c) = \sum [2(a^4 + b^4) + 10a^2 b^2 - 7ab(a^2 + b^2)]
\]

\[
= \sum (a-b)^2 (2a^2 - 3ab + 2b^2) \geq 0.
\]

The equality occurs for \( a = b = c \).

**Second Solution.** Write the inequality in P 1.81 as

\[
\sum \left[ 2 - \frac{2a^2 + (2k+1)bc}{b^2 + kbc + c^2} \right] \leq \frac{3}{k+2},
\]

\[
\sum \frac{2(b^2 + c^2) - bc - 2a^2}{b^2 + kbc + c^2} \leq \frac{3}{k+2}.
\]

Since \( b^2 + c^2 \geq 2bc \), we get

\[
\sum \frac{3bc - 2a^2}{b^2 + kbc + c^2} \leq \frac{3}{k+2},
\]

which is just the desired inequality.

\[
\square
\]

**P 1.83.** If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then

\[
\frac{a^2 + 16bc}{b^2 + c^2} + \frac{b^2 + 16ca}{c^2 + a^2} + \frac{c^2 + 16ab}{a^2 + b^2} \geq 10.
\]

(Vasile Cirtoaje, 2005)
Solution. Let \( a \leq b \leq c \), and

\[
E(a, b, c) = \frac{a^2 + 16bc}{b^2 + c^2} + \frac{b^2 + 16ca}{c^2 + a^2} + \frac{c^2 + 16ab}{a^2 + b^2}.
\]

Consider two cases.

Case 1: \( 16b^3 \geq ac^2 \). We will show that

\[
E(a, b, c) \geq E(0, b, c) \geq 10.
\]

We have

\[
E(a, b, c) - E(0, b, c) = \frac{a^2}{b^2 + c^2} + \frac{a(16c^3 - ab^2)}{c^2(c^2 + a^2)} + \frac{a(16b^3 - ac^2)}{b^2(a^2 + b^2)} \geq 0,
\]

since \( c^3 - ab^2 \geq 0 \) and \( 16b^3 - ac^2 \geq 0 \). Also,

\[
E(0, b, c) - 10 = \frac{16bc}{b^2 + c^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} - 10 = \frac{(b - c)^4(b^2 + c^2 + 4bc)}{b^2c^2(b^2 + c^2)} \geq 0.
\]

Case 2: \( 16b^3 \leq ac^2 \). It suffices to show that

\[
\frac{c^2 + 16ab}{a^2 + b^2} \geq 10.
\]

Indeed,

\[
\frac{c^2 + 16ab}{a^2 + b^2} - 10 \geq \frac{16b^3}{a} + \frac{16ab}{a^2 + b^2} - 10 = \frac{16b}{a} - 10 \geq 16 - 10 > 0.
\]

This completes the proof. The equality holds for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\( \square \)

**P 1.84.** If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then

\[
\frac{a^2 + 128bc}{b^2 + c^2} + \frac{b^2 + 128ca}{c^2 + a^2} + \frac{c^2 + 128ab}{a^2 + b^2} \geq 46.
\]

*(Vasile Cîrtoaje, 2005)*
Solution. Let \(a \leq b \leq c\), and
\[
E(a, b, c) = \frac{a^2 + 128bc}{b^2 + c^2} + \frac{b^2 + 128ca}{c^2 + a^2} + \frac{c^2 + 128ab}{a^2 + b^2}.
\]
Consider two cases.

Case 1: \(128b^3 \geq ac^2\). We will show that
\[
E(a, b, c) \geq E(0, b, c) \geq 46.
\]
We have
\[
E(a, b, c) - E(0, b, c) = \frac{a^2}{b^2 + c^2} + \frac{a(128c^3 - ab^2)}{c^2(c^2 + a^2)} + \frac{a(128b^3 - ac^2)}{b^2(a^2 + b^2)} \geq 0,
\]
since \(c^3 - ab^2 \geq 0\) and \(128b^3 - ac^2 \geq 0\). Also,
\[
E(0, b, c) - 46 = \frac{128bc}{b^2 + c^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} - 46
\]
\[
= \frac{(b^2 + c^2 - 4bc)^2(b^2 + c^2 + 8bc)}{b^2c^2(b^2 + c^2)} \geq 0.
\]

Case 2: \(128b^3 \leq ac^2\). It suffices to show that
\[
\frac{c^2 + 128ab}{a^2 + b^2} \geq 46.
\]
Indeed,
\[
\frac{c^2 + 128ab}{a^2 + b^2} - 46 \geq \frac{128b^3}{a} + \frac{128ab}{a^2 + b^2} - 46
\]
\[
= \frac{128b}{a} - 46 \geq 128 - 46 > 0.
\]
This completes the proof. The equality holds for \(a = 0\) and \(\frac{b}{c} + \frac{c}{b} = 4\) (or any cyclic permutation).

\[\square\]

P 1.85. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{a^2 + 64bc}{(b+c)^2} + \frac{b^2 + 64ca}{(c+a)^2} + \frac{c^2 + 64ab}{(a+b)^2} \geq 18.
\]

(Vasile Cirtoaje, 2005)
Solution. Let \( a \leq b \leq c \), and
\[
E(a, b, c) = \frac{a^2 + 64bc}{(b + c)^2} + \frac{b^2 + 64ca}{(c + a)^2} + \frac{c^2 + 64ab}{(a + b)^2}.
\]
Consider two cases.

Case 1: \( 64b^3 \geq c^2(a + 2b) \). We will show that
\[
E(a, b, c) \geq E(0, b, c) \geq 18.
\]
We have
\[
E(a, b, c) - E(0, b, c) = \frac{a^2}{(b + c)^2} + \frac{a[64c^3 - b^2(a + 2c)]}{c^2(c + a)^2} + \frac{a[64b^3 - c^2(a + 2b)]}{b^2(a + b)^2} \\
\geq \frac{a[64c^3 - b^2(a + 2c)]}{c^2(c + a)^2} \geq \frac{a[64b^2c - b^2(c + 2c)]}{c^2(c + a)^2} = \frac{61ab^2c}{c^2(c + a)^2} \geq 0.
\]
Also,
\[
E(0, b, c) - 18 = \frac{64bc}{(b + c)^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} - 18 \\
= \frac{(b - c)^4(b^2 + c^2 + 6bc)}{b^2c^2(b + c)^2} \geq 0.
\]

Case 2: \( 64b^3 \leq c^2(a + 2b) \). It suffices to show that
\[
\frac{c^2 + 64ab}{(a + b)^2} \geq 18.
\]
Indeed,
\[
\frac{c^2 + 64ab}{(a + b)^2} - 18 \geq \frac{64b^3}{a + 2b} + 64ab \\
\geq \frac{64b^3}{a + 2b} - 18 \geq \frac{64}{3} - 18 > 0.
\]
This completes the proof. The equality holds for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[\square\]

P 1.86. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If \( k \geq -1 \), then
\[
\sum \frac{a^2(b + c) + kabc}{b^2 + kbc + c^2} \geq a + b + c.
\]
Solution. We apply the SOS method. Write the inequality as follows

\[
\sum \left[ \frac{a^2(b + c) + kabc}{b^2 + kbc + c^2} - a \right] \geq 0,
\]

\[
\sum \frac{a(ab + ac - b^2 - c^2)}{b^2 + kbc + c^2} \geq 0,
\]

\[
\sum \frac{ab(a - b)}{b^2 + kbc + c^2} + \sum \frac{ac(a - c)}{b^2 + kbc + c^2} \geq 0,
\]

\[
\sum \frac{ab(a - b)}{b^2 + kbc + c^2} + \sum \frac{ba(b - a)}{c^2 + kca + a^2} \geq 0,
\]

\[
\sum ab(a^2 + kab + b^2)(a + b + kc)(a - b)^2 \geq 0.
\]

Without loss of generality, assume that \(a \geq b \geq c\). Since \(a + b + kc \geq a + b - c > 0\), it suffices to show that

\[
b(b^2 + kbc + c^2)(b + c + ka)(b - c)^2 + a(c^2 + kca + a^2)(c + a + kb)(c - a)^2 \geq 0.
\]

Since \(c + a + kb \geq c + a - b \geq 0\) and \(c^2 + kca + a^2 \geq b^2 + kbc + c^2\), it is enough to prove that

\[
b(b + c + ka)(b - c)^2 + a(c + a + kb)(c - a)^2 \geq 0.
\]

We have

\[
b(b + c + ka)(b - c)^2 + a(c + a + kb)(c - a)^2 \geq
\]

\[
\geq [b(b + c + ka) + a(c + a + kb)](b - c)^2
\]

\[
= [a^2 + b^2 + 2kab + c(a + b)](b - c)^2
\]

\[
\geq [(a - b)^2 + c(a + b)](b - c)^2 \geq 0.
\]

The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\( \square \)

P 1.87. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(k \geq -\frac{3}{2}\), then

\[
\sum \frac{a^3 + (k + 1)abc}{b^2 + kbc + c^2} \geq a + b + c.
\]

(Vasile Cîrtoaje, 2009)
Since it is enough to show that any cyclic permutation).

This completes the proof. The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).
P 1.88. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. If $k > 0$, then

$$\frac{2a^k - b^k - c^k}{b^2 - bc + c^2} + \frac{2b^k - c^k - a^k}{c^2 - ca + a^2} + \frac{2c^k - a^k - b^k}{a^2 - ab + b^2} \geq 0.$$  

(Vasile Cîrtoaje, 2004)

Solution. Let

$$X = b^k - c^k, \quad Y = c^k - a^k, \quad Z = a^k - b^k,$$

$$A = b^2 - bc + c^2, \quad B = c^2 - ca + a^2, \quad C = a^2 - ab + b^2.$$  

Without loss of generality, assume that $a \geq b \geq c$. This involves $A \leq B, A \leq C, X \geq 0$, and $Z \geq 0$. Since

$$\sum_{cyc} \frac{2a^k - b^k - c^k}{b^2 - bc + c^2} = \frac{X + 2Z}{A} + \frac{X - Z}{B} - \frac{2X + Z}{C} = X \left( \frac{1}{A} + \frac{1}{B} - \frac{2}{C} \right) + Z \left( \frac{2}{A} - \frac{1}{B} - \frac{1}{C} \right),$$

it suffices to prove that

$$\frac{1}{A} + \frac{1}{B} - \frac{2}{C} \geq 0.$$  

Write this inequality as

$$\frac{1}{A} - \frac{1}{C} \geq \frac{1}{C} - \frac{1}{B},$$

that is,

$$(a - c)(a + c - b)(a^2 - ac + c^2) \geq (b - c)(a - b - c)(b^2 - bc + c^2).$$

For the nontrivial case $a > b + c$, we can obtain this inequality from

$$a + c - b \geq a - b - c,$$

$$a - c \geq b - c,$$

$$a^2 - ac + c^2 > b^2 - bc + c^2.$$  

This completes the proof. The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). \qed
P 1.89. If \( a, b, c \) are the lengths of the sides of a triangle, then

(a) \[
\frac{b + c - a}{b^2 - bc + c^2} + \frac{c + a - b}{c^2 - ca + a^2} + \frac{a + b - c}{a^2 - ab + b^2} \geq \frac{2(a + b + c)}{a^2 + b^2 + c^2};
\]

(b) \[
\frac{a^2 - 2bc}{b^2 - bc + c^2} + \frac{b^2 - 2ca}{c^2 - ca + a^2} + \frac{c^2 - 2ab}{a^2 - ab + b^2} \leq 0.
\]

\text{(Vasile Cîrtoaje, 2009)}

\textbf{Solution.} (a) By the Cauchy-Schwarz inequality, we get

\[
\sum \frac{b + c - a}{b^2 - bc + c^2} \geq \frac{[\sum (b + c - a)]^2}{\sum (b + c - a)(b^2 - bc + c^2)} = \frac{(\sum a)^2}{2 \sum a^3 - \sum a^2(b + c) + 3abc}.
\]

On the other hand, from

\[
(b + c - a)(c + a - b)(a + b - c) \geq 0,
\]

we get

\[2abc \leq \sum a^2(b + c) - \sum a^3,
\]

and hence

\[2 \sum a^3 - \sum a^2(b + c) + 3abc \leq \frac{\sum a^3 + \sum a^2(b + c)}{2} = \frac{(\sum a)(\sum a^2)}{2}.
\]

Therefore,

\[\sum \frac{b + c - a}{b^2 - bc + c^2} \geq \frac{2 \sum a}{\sum a^2}.
\]

The equality holds for a degenerate triangle with \( a = b + c \) (or any cyclic permutation).

(b) Since

\[
\sum \frac{a^2 - 2bc}{b^2 - bc + c^2} = 2 - \frac{(b - c)^2 + (b + c)^2 - a^2}{b^2 - bc + c^2},
\]

we can write the inequality as

\[\sum \frac{(b - c)^2}{b^2 - bc + c^2} + (a + b + c) \sum \frac{b + c - a}{b^2 - bc + c^2} \geq 6.
\]

Using the inequality in (a), it suffices to prove that

\[\sum \frac{(b - c)^2}{b^2 - bc + c^2} + \frac{2(a + b + c)^2}{a^2 + b^2 + c^2} \geq 6.
\]
Write this inequality as
\[
\sum \frac{(b-c)^2}{b^2 - bc + c^2} \geq \sum \frac{2(b-c)^2}{a^2 + b^2 + c^2},
\]
or, equivalently,
\[
\sum \frac{(b-c)^2(a-b+c)(a+b-c)}{b^2 - bc + c^2} \geq 0,
\]
which is true. The equality holds for degenerate triangles with either \(a/2 = b = c\) (or any cyclic permutation), or \(a = 0\) and \(b = c\) (or any cyclic permutation).

**Remark.** The following generalization of the inequality in (b) holds (Vasile Cîrtoaje, 2009):

- Let \(a, b, c\) be the lengths of the sides of a triangle. If \(k \geq -1\), then
  \[
  \sum \frac{a^2 - 2(k + 2)bc}{b^2 + kbc + c^2} \leq 0.
  \]
with equality for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\(\square\)

**P 1.90.** If \(a, b, c\) are nonnegative real numbers, then
\[
\frac{a^2}{5a^2 + (b + c)^2} + \frac{b^2}{5b^2 + (c + a)^2} + \frac{c^2}{5c^2 + (a + b)^2} \leq \frac{1}{3}.
\]
(Vo Quoc Ba Can, 2009)

**Solution.** Apply the Cauchy-Schwarz inequality in the following manner
\[
\frac{9}{5a^2 + (b + c)^2} = \frac{(1 + 2)^2}{(a^2 + b^2 + c^2) + 2(2a^2 + bc)} \leq \frac{1}{a^2 + b^2 + c^2} + \frac{2}{2a^2 + bc}.
\]
Then,
\[
\sum \frac{9a^2}{5a^2 + (b + c)^2} \leq \sum \frac{a^2}{a^2 + b^2 + c^2} + \sum \frac{2a^2}{2a^2 + bc} = 4 - \sum \frac{bc}{2a^2 + bc},
\]
and it remains to show that
\[
\sum \frac{bc}{2a^2 + bc} \geq 1.
\]
This is a known inequality, which can be proved by the Cauchy-Schwarz inequality, as follows
\[
\sum \frac{bc}{2a^2 + bc} \geq \frac{(\sum bc)^2}{\sum bc(2a^2 + bc)} = 1.
\]
The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\(\square\)
P 1.91. If $a, b, c$ are nonnegative real numbers, then

$$\frac{b^2 + c^2 - a^2}{2a^2 + (b + c)^2} + \frac{c^2 + a^2 - b^2}{2b^2 + (c + a)^2} + \frac{a^2 + b^2 - c^2}{2c^2 + (a + b)^2} \geq \frac{1}{2}.$$  

(Vasile Cîrtoaje, 2011)

Solution. We apply the SOS method. Write the inequality as follows

$$\sum \left[ \frac{b^2 + c^2 - a^2}{2a^2 + (b + c)^2} - \frac{1}{6} \right] \geq 0,$$

$$\sum \frac{5(b^2 + c^2 - 2a^2) + 2(a^2 - bc)}{2a^2 + (b + c)^2} \geq 0;$$

$$\sum \frac{5(b^2 - a^2) + 5(c^2 - a^2) + (a - b)(a + c) + (a - c)(a + b)}{2a^2 + (b + c)^2} \geq 0;$$

$$\sum \frac{(b - a)[5(b + a) - (a + c)]}{2a^2 + (b + c)^2} + \sum \frac{(c - a)[5(c + a) - (a + b)]}{2a^2 + (b + c)^2} \geq 0;$$

$$\sum \frac{(b - a)[5(b + a) - (a + c)]}{2a^2 + (b + c)^2} + \sum \frac{(a - b)[5(a + b) - (b + c)]}{2b^2 + (c + a)^2} \geq 0;$$

$$\sum (a - b)^2 [2c^2 + (a + b)^2] \left[ 2(a^2 + b^2) + c^2 + 3ab - 3c(a + b) \right] \geq 0,$$

$$\sum (b - c)^2 R_a S_a \geq 0,$$

where

$$R_a = 2a^2 + (b + c)^2, \quad S_a = a^2 + 2(b^2 + c^2) + 3bc - 3a(b + c).$$

Without loss of generality, assume that $a \geq b \geq c$. We have

$$S_b = b^2 + 2(c^2 + a^2) + 3ca - 3b(c + a) = (a - b)(2a - b) + 2c^2 + 3c(a - b) \geq 0,$$

$$S_c = c^2 + 2(a^2 + b^2) + 3ab - 3c(a + b) \geq 7ab - 3c(a + b) \geq 3a(b - c) + 3b(a - c) \geq 0,$$

$$S_a + S_b = 3(a - b)^2 + 4c^2 \geq 0.$$

Since

$$\sum (b - c)^2 R_a S_a \geq (b - c)^2 R_a S_a + (c - a)^2 R_b S_b$$

$$= (b - c)^2 R_a (S_a + S_b) + [(c - a)^2 R_b - (b - c)^2 R_a] S_b,$$

it suffices to prove that

$$(a - c)^2 R_b \geq (b - c)^2 R_a.$$

We can get this by multiplying the inequalities

$$b^2 (a - c)^2 \geq a^2 (b - c)^2$$

and

$$(a - c)^2 R_b \geq (b - c)^2 R_a.$$
and
\[ a^2R_b \geq b^2R_a. \]
The equality holds for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation). \( \square \)

**P 1.92.** Let \( a, b, c \) be positive real numbers. If \( k > 0 \), then
\[
\frac{3a^2 - 2bc}{ka^2 + (b - c)^2} + \frac{3b^2 - 2ca}{kb^2 + (c - a)^2} + \frac{3c^2 - 2ab}{kc^2 + (a - b)^2} \leq \frac{3}{k}.
\]
*(Vasile Cîrtoaje, 2011)*

**Solution.** Use the SOS method. Write the inequality as follows

\[
\sum \left[ \frac{1}{k} \left( \frac{3a^2 - 2bc}{ka^2 + (b - c)^2} \right) \right] \geq 0,
\]
\[
\sum \frac{b^2 + c^2 - 2a^2 + 2(k-1)(bc - a^2)}{ka^2 + (b - c)^2} \geq 0;
\]
\[
\sum \frac{(b^2 - a^2) + (c^2 - a^2) + (k-1)[(a+b)(c-a) + (a+c)(b-a)]}{ka^2 + (b - c)^2} \geq 0;
\]
\[
\sum \frac{(b-a)[b + a + (k-1)(a + c)]}{ka^2 + (b - c)^2} + \sum \frac{(c-a)[c + a + (k-1)(a + b)]}{ka^2 + (b - c)^2} \geq 0;
\]
\[
\sum \frac{(b-a)[b + a + (k-1)(a + c)]}{ka^2 + (b - c)^2} + \sum \frac{(a-b)[a + b + (k-1)(b + c)]}{kb^2 + (c - a)^2} \geq 0;
\]
\[
\sum (a-b)^2[kc^2 + (a-b)^2][(k-1)c^2 + 2c(a+b) + (k^2-1)(ab + bc + ca)] \geq 0.
\]

For \( k \geq 1 \), the inequality is clearly true. Consider further that \( 0 < k < 1 \). Since
\[
(k-1)c^2 + 2c(a+b) + (k^2-1)(ab + bc + ca) >
\]
\[
> -c^2 + 2c(a+b) - (ab + bc + ca) = (b-c)(c-a),
\]
it suffices to prove that
\[
(a-b)(b-c)(c-a) \sum (a-b)[kc^2 + (a-b)^2] \geq 0.
\]
Since
\[
\sum (a-b)[kc^2 + (a-b)^2] = k \sum (a-b)c^2 + \sum (a-b)^3
\]
\[
= (3-k)(a-b)(b-c)(c-a),
\]
we have
\[(a - b)(b - c)(c - a) \sum (a - b)[kc^2 + (a - b)^2] =
= (3 - k)(a - b)^2(b - c)^2(c - a)^2 \geq 0.\]
This completes the proof. The equality holds for \(a = b = c\). $
\square$

**P 1.93.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(k \geq 3 + \sqrt{7}\), then

(a) \[\frac{a}{a^2 + kbc} + \frac{b}{b^2 + kca} + \frac{c}{c^2 + kab} \geq \frac{9}{(1 + k)(a + b + c)};\]

(b) \[\frac{1}{ka^2 + bc} + \frac{1}{kb^2 + ca} + \frac{1}{kc^2 + ab} \geq \frac{9}{(k + 1)(ab + bc + ca)}.\]

(Vasile Cîrtoaje, 2005)

**Solution.** (a) Assume that \(a = \max\{a, b, c\}\). Setting \(t = (b + c)/2, t \leq a\), by the Cauchy-Schwarz inequality, we get

\[\frac{b}{b^2 + kca} + \frac{c}{c^2 + kab} \geq \frac{(b + c)^2}{b(b^2 + kca) + c(c^2 + kab)} = \frac{4t^2}{b^2 + 2t^2 + (ka - 3t)bc} \geq \frac{2t^2}{4t^2 + (ka - 3t)t^2} = \frac{2}{t + ka}.\]

On the other hand,
\[\frac{a}{a^2 + kbc} \geq \frac{a}{a^2 + kt^2}.\]
Therefore, it suffices to prove that
\[\frac{a}{a^2 + kt^2} + \frac{2}{t + ka} \geq \frac{9}{(k + 1)(a + 2t)},\]
which is equivalent to
\[(a - t)^2[(k^2 - 6k + 2)a + k(4k - 5)t] \geq 0.\]
This inequality is true, since \(k^2 - 6k + 2 \geq 0\) and \(4k - 5 > 0\). The equality holds for \(a = b = c\).

(b) For \(a = 0\), the inequality becomes
\[\frac{1}{b^2} + \frac{1}{c^2} \geq \frac{k(8 - k)}{(k + 1)bc}.\]
We have
\[
\frac{1}{b^2} + \frac{1}{c^2} - \frac{k(8-k)}{(k+1)bc} \geq \frac{2}{bc} - \frac{k(8-k)}{(k+1)bc} = \frac{k^2 - 6k + 2}{(k+1)bc} \geq 0.
\]
For \(a, b, c > 0\), the desired inequality follows from the inequality in (a) by substituting \(a, b, c\) with \(1/a, 1/b, 1/c\), respectively. The equality holds for \(a = b = c\). In the case \(k = 3 + \sqrt{7}\), the equality also holds for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.94.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq \frac{6}{a^2 + b^2 + c^2 + ab + bc + ca}.
\]

*(Vasile Cîrtoaje, 2005)*

**Solution.** Applying the Cauchy-Schwarz inequality, we have
\[
\sum \frac{1}{2a^2 + bc} = \sum \frac{(b + c)^2}{(b + c)^2(2a^2 + bc)} \geq \frac{4(a + b + c)^2}{\sum (b + c)^2(2a^2 + bc)}.
\]
Thus, it suffices to show that
\[
2(a + b + c)^2(a^2 + b^2 + c^2 + ab + bc + ca) \geq 3 \sum (b + c)^2(2a^2 + bc),
\]
which is equivalent to
\[
2 \sum a^4 + 3 \sum ab(a^2 + b^2) + 2abc \sum a \geq 10 \sum a^2b^2.
\]
This follows by adding Schur’s inequality
\[
2 \sum a^4 + 2abc \sum a \geq 2 \sum ab(a^2 + b^2)
\]
to the inequality
\[
5 \sum ab(a^2 + b^2) \geq 10 \sum a^2b^2.
\]
The equality holds for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.95.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{22a^2 + 5bc} + \frac{1}{22b^2 + 5ca} + \frac{1}{22c^2 + 5ab} \geq \frac{1}{(a + b + c)^2}.
\]

*(Vasile Cîrtoaje, 2005)*
**Solution.** Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{2a^2 + 5bc} = \sum \frac{(b + c)^2}{(b + c)^2(2a^2 + 5bc)} \geq \frac{4(a + b + c)^2}{\sum (b + c)^2(2a^2 + 5bc)}.$$  

Thus, it suffices to show that

$$4(a + b + c)^4 \geq \sum (b + c)^2(22a^2 + 5bc),$$

which is equivalent to

$$4 \sum a^4 + 11 \sum ab(a^2 + b^2) + 4abc \sum a \geq 30 \sum a^2 b^2.$$

This follows by adding Schur's inequality

$$4 \sum a^4 + 4abc \sum a \geq 4 \sum ab(a^2 + b^2)$$

to the inequality

$$15 \sum ab(a^2 + b^2) \geq 30 \sum a^2 b^2.$$  

The equality holds for $a = b = c$.

\[\square\]

**P 1.96.** Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq \frac{8}{(a + b + c)^2}.$$  

*(Vasile Cîrtoaje, 2005)*

**First Solution.** Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{2a^2 + bc} = \sum \frac{(b + c)^2}{(b + c)^2(2a^2 + 5bc)} \geq \frac{4(a + b + c)^2}{\sum (b + c)^2(22a^2 + 5bc)}.$$  

Thus, it suffices to show that

$$(a + b + c)^4 \geq 2 \sum (b + c)^2(2a^2 + bc),$$

which is equivalent to

$$\sum a^4 + 2 \sum ab(a^2 + b^2) + 4abc \sum a \geq 6 \sum a^2 b^2.$$  

We will prove the sharper inequality

$$\sum a^4 + 2 \sum ab(a^2 + b^2) + abc \sum a \geq 6 \sum a^2 b^2.$$
This follows by adding Schur’s inequality
\[ \sum a^4 + abc \sum a \geq \sum ab(a^2 + b^2) \]
to the inequality
\[ 3 \sum ab(a^2 + b^2) \geq 6 \sum a^2 b. \]
The equality holds for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

**Second Solution.** Without loss of generality, we may assume that \( a \geq b \geq c \). Since the equality holds for \( c = 0 \) and \( a = b \), write the inequality as
\[
\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{4c^2 + 2ab} + \frac{1}{4c^2 + 2ab} \geq \frac{8}{(a + b + c)^2}
\]
and then apply the Cauchy-Schwarz inequality. It suffices to prove that
\[
\frac{16}{(2a^2 + bc) + (2b^2 + ca) + (4c^2 + 2ab) + (4c^2 + 2ab)} \geq \frac{8}{(a + b + c)^2},
\]
which is equivalent to the obvious inequality
\[ c(a + b - 2c) \geq 0. \]

\[ \square \]

**P 1.97.** Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that
\[
\frac{1}{a^2 + bc} + \frac{1}{b^2 + ca} + \frac{1}{c^2 + ab} \geq \frac{12}{(a + b + c)^2}.
\]
*(Vasile Cirtoaje, 2005)*

**Solution.** Due to homogeneity, we may assume that \( a + b + c = 1 \). On this assumption, write the inequality as
\[
\sum \left( \frac{1}{a^2 + bc} - 1 \right) \geq 9,
\]
\[
\sum \frac{1 - a^2 - bc}{a^2 + bc} \geq 9.
\]
Since
\[ 1 - a^2 - bc = (a + b + c)^2 - a^2 - bc > 0, \]
by the Cauchy-Schwarz inequality, we have
\[
\sum \frac{1 - a^2 - bc}{a^2 + bc} \geq \frac{[\sum (1 - a^2 - bc)]^2}{\sum (1 - a^2 - bc)(a^2 + bc)}.
\]
Then, it suffices to prove that
\[(3 - \sum a^2 - \sum bc)^2 \geq 9 \sum (a^2 + bc) - 9 \sum (a^2 + bc)^2,\]
which is equivalent to
\[(1 - 4q)(4 - 7q) + 36abc \geq 0,\]
where \(q = ab + bc + ca\). For \(q \leq 1/4\), this inequality is clearly true. Consider further that \(q > 1/4\). By Schur’s inequality of degree three
\[(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),\]
we get \(1 + 9abc \geq 4q\), and hence \(36abc \geq 16q - 4\). Thus,
\[(1 - 4q)(4 - 7q) + 36abc \geq (1 - 4q)(4 - 7q) + 16q - 4 = 7q(4q - 1) > 0.\]
The equality holds for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\Box\]

**P 1.98.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

(a) \(\frac{1}{a^2 + 2bc} + \frac{1}{b^2 + 2ca} + \frac{1}{c^2 + 2ab} \geq \frac{1}{a^2 + b^2 + c^2} + \frac{2}{ab + bc + ca}^{-};\)

(b) \(\frac{a(b + c)}{a^2 + 2bc} + \frac{b(c + a)}{b^2 + 2ca} + \frac{c(a + b)}{c^2 + 2ab} \geq 1 + \frac{ab + bc + ca}{a^2 + b^2 + c^2}.\)

(Darij Grinberg and Vasile Cirtoaje, 2005)

**Solution.** (a) Write the inequality as
\[\frac{\sum (b^2 + 2ca)(c^2 + 2ab)}{(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab)} \geq \frac{ab + bc + ca + 2a^2 + 2b^2 + 2c^2}{(a^2 + b^2 + c^2)(ab + bc + ca)}.\]

Since
\[\sum (b^2 + 2ca)(c^2 + 2ab) = (ab + bc + ca)(ab + bc + ca + 2a^2 + 2b^2 + 2c^2),\]
it suffices to show that
\[(a^2 + b^2 + c^2)(ab + bc + ca)^2 \geq (a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab),\]
which is just the inequality (a) in P 1.16 in Volume 1. The equality holds for \(a = b\), or \(b = c\), or \(c = a\).
(b) Write the inequality in (a) as
\[ \sum \frac{ab + bc + ca}{a^2 + 2bc} \geq 2 + \frac{ab + bc + ca}{a^2 + b^2 + c^2}, \]
or
\[ \sum \frac{ab + bc + ca}{a^2 + 2bc} \geq 2 + \frac{ab + bc + ca}{a^2 + b^2 + c^2}. \]
The desired inequality follows by adding this inequality to
\[ 1 \geq \sum \frac{bc}{a^2 + 2bc}. \]
The last inequality is equivalent to
\[ \sum \frac{a^2}{a^2 + 2bc} \geq 1, \]
which follows by applying the AM-GM inequality as follows
\[ \sum \frac{a^2}{a^2 + 2bc} \geq \sum \frac{a}{a^2 + b^2 + c^2} = 1. \]
The equality holds for \( a = b = c \).

\[ \square \]

**P 1.99.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

(a) \[ \frac{a}{a^2 + 2bc} + \frac{b}{b^2 + 2ca} + \frac{c}{c^2 + 2ab} \leq \frac{a + b + c}{ab + bc + ca}; \]

(b) \[ \frac{ab + bc + ca}{a^2 + 2bc} + \frac{c(a + b)}{b^2 + 2ca} + \frac{c^2 + 2ab}{2ab} \leq 1 + \frac{a^2 + b^2 + c^2}{ab + bc + ca}. \]

*(Vasile Cirtoaje, 2008)*

**Solution.** (a) Use the SOS method. Write the inequality as
\[ \sum a \left( 1 - \frac{ab + bc + ca}{a^2 + 2bc} \right) \geq 0, \]
and
\[ \sum \frac{a(a - b)(a - c)}{a^2 + 2bc} \geq 0. \]
Assume that \( a \geq b \geq c \). Since \( (c - a)(c - b) \geq 0 \), it suffices to show that
\[ \frac{a(a - b)(a - c)}{a^2 + 2bc} + \frac{b(b - a)(b - c)}{b^2 + 2ca} \geq 0. \]
This inequality is equivalent to
\[ c(a - b)^2[2a(a - c) + 2b(b - c) + 3ab] \geq 0, \]
which is clearly true. The equality holds for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

(b) Since
\[ \frac{a(b + c)}{a^2 + 2bc} = \frac{a(a + b + c)}{a^2 + 2bc} - \frac{a^2}{a^2 + 2bc}, \]
we can write the inequality as
\[ (a + b + c)\sum \frac{a}{a^2 + 2bc} \leq 1 + \frac{a^2 + b^2 + c^2}{ab + bc + ca} + \sum \frac{a^2}{a^2 + 2bc}. \]
According to the inequality in (a), it suffices to show that
\[ \frac{(a + b + c)^2}{ab + bc + ca} \leq 1 + \frac{a^2 + b^2 + c^2}{ab + bc + ca} + \sum \frac{a^2}{a^2 + 2bc}, \]
which is equivalent to
\[ \sum \frac{a^2}{a^2 + 2bc} \geq 1. \]
Indeed,
\[ \sum \frac{a^2}{a^2 + 2bc} \geq \sum \frac{a^2}{a^2 + b^2 + c^2} = 1. \]
The equality holds for \( a = b = c \).

□

**P 1.100.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

(a) \[ \frac{a}{2a^2 + bc} + \frac{b}{2b^2 + ca} + \frac{c}{2c^2 + ab} \geq \frac{a + b + c}{a^2 + b^2 + c^2}; \]

(b) \[ \frac{b + c}{2a^2 + bc} + \frac{c + a}{2b^2 + ca} + \frac{a + b}{2c^2 + ab} \geq \frac{6}{a + b + c}. \]

*(Vasile Cîrtoaje, 2008)*

**Solution.** Assume that \( a \geq b \geq c \).

(a) Multiplying by \( a + b + c \), we can write the inequality as follows
\[ \sum \frac{a(a + b + c)}{2a^2 + bc} \geq \frac{(a + b + c)^2}{a^2 + b^2 + c^2}, \]
Also, it suffices to show that $f(a, b, c)$ holds for $a = b = c$, where

$$f(a, b, c) = \sum a^2(a - b)(a - c) \geq 2a^2 + bc,$$

and for $a = 0$ and $b = c$ (or any cyclic permutation).

(b) We apply the SOS method. Write the inequality as follows

$$\sum \left[ \frac{(b + c)(a + b + c)}{2a^2 + bc} - 2 \right] \geq 0,$$

$$\sum \left( \frac{b^2 + ab - 2a^2}{2a^2 + bc} + \frac{c^2 + ca - 2a^2}{2a^2 + bc} \right) \geq 0,$$

$$\sum \left( \frac{(b - a)(b + 2a) + (c - a)(c + 2a)}{2a^2 + bc} \right) \geq 0,$$

$$\sum \left( \frac{(b - a)(b + 2a) + (a - b)(a + 2b)}{2a^2 + bc} \right) \geq 0.$$
\[(a-b)\left(\frac{a+2b}{2b^2+ca} - \frac{b+2a}{2a^2+bc}\right) \geq 0,\]
\[\sum (a-b)^2(2c^2+ab)(a^2+b^2+3ab-ac-bc) \geq 0.\]

Since \[a^2+b^2+3ab-ac-bc \geq a^2+b^2+2ab-ac-bc = (a+b)(a+b-c),\]
it suffices to show that
\[\sum (a-b)^2(2c^2+ab)(a+b)(a+b-c) \geq 0.\]

This inequality is true if
\[(b-c)^2(2a^2+bc)(b+c)(b+c-a) + (c-a)^2(2b^2+ca)(c+a)(c+a-b) \geq 0;\]
that is,
\[(a-c)^2(2b^2+ca)(a+c)(a+c-b) \geq (b-c)^2(2a^2+bc)(b+c)(a-b-c).\]

Since \(a+c \geq b+c\) and \(a+c-b \geq a-b-c\), it is enough to prove that
\[(a-c)^2(2b^2+ca) \geq (b-c)^2(2a^2+bc).\]

We can obtain this inequality by multiplying the inequalities
\[b^2(a-c)^2 \geq a^2(b-c)^2\]
and
\[a^2(2b^2+ca) \geq b^2(2a^2+bc).\]

The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\Box\]

**P 1.101.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that
\[\frac{a(b+c)}{a^2+bc} + \frac{b(c+a)}{b^2+ca} + \frac{c(a+b)}{c^2+ab} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2}.\]

*(Pham Huu Duc, 2006)*
Solution. Assume that \( a \geq b \geq c \) and write the inequality as follows

\[
3 - \frac{(a + b + c)^2}{a^2 + b^2 + c^2} \geq \sum \left( 1 - \frac{ab + ac}{a^2 + bc} \right),
\]

\[
2 \sum (a - b)(a - c) \geq (a^2 + b^2 + c^2) \sum \frac{(a - b)(a - c)}{a^2 + bc},
\]

\[
\sum \frac{(a - b)(a - c)(a + b - c)(a - b + c)}{a^2 + bc} \geq 0.
\]

It suffices to show that

\[
\frac{(b - c)(b - a)(b + c - a)(b - c + a)}{b^2 + ca} + \frac{(c - a)(c - b)(c + a - b)(c - a + b)}{c^2 + ab} \geq 0,
\]

which is equivalent to the obvious inequality

\[
\frac{(b - c)^2(c - a + b)^2(a + bc)}{(b^2 + ca)(c^2 + ab)} \geq 0.
\]

The equality holds for \( a = b = c \), and for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[\square\]

P 1.102. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If \( k > 0 \), then

\[
\frac{b^2 + c^2 + \sqrt{3}bc}{a^2 + kbc} + \frac{c^2 + a^2 + \sqrt{3}ca}{b^2 + kca} + \frac{a^2 + b^2 + \sqrt{3}ab}{c^2 + kab} \geq \frac{3(2 + \sqrt{3})}{1 + k}.
\]

(Vasile Cîrtoaje, 2013)

Solution. Write the inequality in the form \( f_6(a, b, c) \geq 0 \), where

\[
f_6(a, b, c) = (1 + k) \sum (b^2 + c^2 + \sqrt{3}bc)(b^2 + kca)(c^2 + kab)
\]

\[-3(2 + \sqrt{3})(a^2 + kbc)(b^2 + kca)(c^2 + kab).
\]

Clearly, \( f_6(a, b, c) \) has the same highest coefficient as \( f(a, b, c) \), where

\[
f(a, b, c) = (1 + k) \sum (\sqrt{3}bc - a^2)(b^2 + kca)(c^2 + kab)
\]

\[-3(2 + \sqrt{3})(a^2 + kbc)(b^2 + kca)(c^2 + kab));
\]

therefore,

\[
A = 3(1 + k)^3(\sqrt{3} - 1) - 3(2 + \sqrt{3})(1 + k)^3
\]

\[= -9(1 + k)^3.\]
Since \( A \leq 0 \), according to P 2.76-(a) in Volume 1, it suffices to prove the original inequality for \( b = c = 1 \) and for \( a = 0 \).

In the first case, this inequality is equivalent to

\[
(a - 1)^2 \left[ (k + 1)a^2 - \left( 1 + \frac{\sqrt{3}}{2} \right) (k - 2)a + \left( k - \frac{1 + \sqrt{3}}{2} \right)^2 \right] \geq 0.
\]

For the nontrivial case \( k > 2 \), we have

\[
(k + 1)a^2 + \left( k - \frac{1 + \sqrt{3}}{2} \right)^2 \geq 2\sqrt{k + 1} \left( k - \frac{1 + \sqrt{3}}{2} \right) a
\]

\[
\geq 2\sqrt{3} \left( k - \frac{1 + \sqrt{3}}{2} \right) a \geq \left( 1 + \frac{\sqrt{3}}{2} \right) (k - 2)a.
\]

In the second case \( (a = 0) \), the original inequality can be written as

\[
\frac{1}{k} \left( \frac{b}{c} + \frac{c}{b} + \sqrt{3} \right) + \left( \frac{b^2}{c^2} + \frac{c^2}{b^2} \right) \geq \frac{3(2 + \sqrt{3})}{1 + k},
\]

and is true if

\[
\frac{2 + \sqrt{3}}{k} + 2 \geq \frac{3(2 + \sqrt{3})}{1 + k},
\]

which is equivalent to

\[
\left( k - \frac{1 + \sqrt{3}}{2} \right)^2 \geq 0.
\]

The equality holds for \( a = b = c \). If \( k = \frac{1 + \sqrt{3}}{2} \), then the equality holds also for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[\square\]

P 1.103. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that

\[
\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{8}{a^2 + b^2 + c^2} \geq \frac{6}{ab + bc + ca}.
\]

(Vasile Cîrtoaje, 2013)

**Solution.** Multiplying by \( a^2 + b^2 + c^2 \), the inequality becomes

\[
\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} + 11 \geq \frac{6(a^2 + b^2 + c^2)}{ab + bc + ca}.
\]
Since
\[
\left( \frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \right) (a^2 b^2 + b^2 c^2 + c^2 a^2) = \]
\[
ar^4 + b^4 + c^4 + a^2 b^2 c^2 \left( \frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \right) \geq a^4 + b^4 + c^4,
\]
it suffices to show that
\[
\frac{a^4 + b^4 + c^4}{a^2 b^2 + b^2 c^2 + c^2 a^2} + 11 \geq \frac{6(a^2 + b^2 + c^2)}{ab + bc + ca},
\]
which is equivalent to
\[
\frac{(a^2 + b^2 + c^2)^2}{a^2 b^2 + b^2 c^2 + c^2 a^2} + 9 \geq \frac{6(a^2 + b^2 + c^2)}{ab + bc + ca}.
\]
Clearly, it is enough to prove that
\[
\left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^2 + 9 \geq \frac{6(a^2 + b^2 + c^2)}{ab + bc + ca},
\]
which is
\[
\left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} - 3 \right)^2 \geq 0.
\]
The equality holds for \(a = 0\) and \(\frac{b}{c} + \frac{c}{b} = 3\) (or any cyclic permutation).

\[\square\]

**P 1.104.** If \(a, b, c\) are the lengths of the sides of a triangle, then
\[
\frac{a(b + c)}{a^2 + 2bc} + \frac{b(c + a)}{b^2 + 2ca} + \frac{c(a + b)}{c^2 + 2ab} \leq 2.
\]

(Vo Quoc Ba Can and Vasile Cirtoaje, 2010)

**Solution.** Write the inequality as
\[
\sum \left( 1 - \frac{ab + ac}{a^2 + 2bc} \right) \geq 1,
\]
\[
\sum \frac{a^2 + 2bc - ab - ac}{a^2 + 2bc} \geq 1.
\]
Since
\[
a^2 + 2bc - ab - ac = bc - (a - c)(b - a) \geq |a - c||b - a| - (a - c)(b - a) \geq 0,
\]
Symmetric Rational Inequalities

by the Cauchy-Schwarz inequality, we have

\[
\sum a^2 + 2bc - ab - ac \geq \frac{[\sum (a^2 + 2bc - ab - ac)]^2}{\sum (a^2 + 2bc)(a^2 + 2bc - ab - ac)}.
\]

Thus, it suffices to prove that

\[
(a^2 + b^2 + c^2)^2 \geq \sum (a^2 + 2bc)(a^2 + 2bc - ab - ac),
\]

which reduces to the obvious inequality

\[
ab(a - b)^2 + bc(b - c)^2 + ca(c - a)^2 \geq 0.
\]

The equality holds for an equilateral triangle, and for a degenerate triangle with \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

P 1.105. If \(a, b, c\) are real numbers, then

\[
\frac{a^2 - bc}{2a^2 + b^2 + c^2} + \frac{b^2 - ca}{2b^2 + c^2 + a^2} + \frac{c^2 - ab}{2c^2 + a^2 + b^2} \geq 0.
\]

(Nguyen Anh Tuan, 2005)

**First Solution.** Rewrite the inequality as

\[
\sum \left( \frac{1}{2} - \frac{a^2 - bc}{2a^2 + b^2 + c^2} \right) \leq \frac{3}{2},
\]

\[
\sum \frac{(b + c)^2}{2a^2 + b^2 + c^2} \leq 3.
\]

If two of \(a, b, c\) are zero, then the inequality is trivial. Otherwise, applying the Cauchy-Schwarz inequality, we get

\[
\sum \frac{(b + c)^2}{2a^2 + b^2 + c^2} = \sum \frac{(b + c)^2}{(a^2 + b^2) + (a^2 + c^2)}
\]

\[
\leq \sum \left( \frac{b^2}{a^2 + b^2} + \frac{c^2}{a^2 + c^2} \right) = \sum \frac{b^2}{a^2 + b^2} + \sum \frac{a^2}{b^2 + a^2} = 3.
\]

The equality holds for \(a = b = c\).

**Second Solution.** Use the SOS method. We have

\[
2 \sum \frac{a^2 - bc}{2a^2 + b^2 + c^2} = \sum \frac{(a - b)(a + c) + (a - c)(a + b)}{2a^2 + b^2 + c^2}
\]
\[
\begin{align*}
150 & \quad \text{Vasile Cîrtoaje} \\
\frac{(a-b)(a+c)}{2a^2+b^2+c^2} + \frac{(b-a)(b+c)}{2b^2+c^2+a^2} \\
&= \sum (a-b) \left( \frac{a+c}{2a^2+b^2+c^2} - \frac{b+c}{2b^2+c^2+a^2} \right) \\
&= (a^2 + b^2 + c^2 - ab - bc - ca) \sum \frac{(a-b)^2}{(2a^2+b^2+c^2)(2b^2+c^2+a^2)} \geq 0.
\end{align*}
\]

P 1.106. If \(a, b, c\) are nonnegative real numbers, then
\[
\frac{3a^2 - bc}{2a^2 + b^2 + c^2} + \frac{3b^2 - ca}{2b^2 + c^2 + a^2} + \frac{3c^2 - ab}{2c^2 + a^2 + b^2} \leq \frac{3}{2},
\]  
(Vasile Cîrtoaje, 2008)

First Solution. Write the inequality as
\[
\sum \left( \frac{3}{2} - \frac{3a^2 - bc}{2a^2 + b^2 + c^2} \right) \geq 3,
\]
\[
\sum \frac{8bc + 3(b-c)^2}{2a^2 + b^2 + c^2} \geq 6.
\]
By the Cauchy-Schwarz inequality, we have
\[
8bc + 3(b-c)^2 \geq \frac{3[4bc + (b-c)^2]^2}{6bc + (b-c)^2} = \frac{2(b+c)^4}{b^2 + c^2 + 4bc}.
\]
Therefore, it suffices to prove that
\[
\sum \frac{(b+c)^4}{(2a^2 + b^2 + c^2)(b^2 + c^2 + 4bc)} \geq 2.
\]
Using again the Cauchy-Schwarz inequality, we get
\[
\sum \frac{(b+c)^4}{(2a^2 + b^2 + c^2)(b^2 + c^2 + 4bc)} \geq \frac{[\sum (b+c)^2]^2}{\sum (2a^2 + b^2 + c^2)(b^2 + c^2 + 4bc)} = 2.
\]
The equality holds for \(a = b = c\), for \(a = 0\) and \(b = c\), and for \(b = c = 0\) (or any cyclic permutation).

Second Solution. Write the inequality as
\[
\sum \left( \frac{1}{2} - \frac{3a^2 - bc}{2a^2 + b^2 + c^2} \right) \geq 0,
\]
\[
\sum \frac{(b + c + 2a)(b + c - 2a)}{2a^2 + b^2 + c^2} \geq 0,
\]
\[
\sum \frac{(b + c + 2a)(b - a) + (b + c + 2a)(c - a)}{2a^2 + b^2 + c^2} \geq 0,
\]
\[
\sum \frac{(b + c + 2a)(b - a)}{2a^2 + b^2 + c^2} + \sum \frac{(a + 2b)(a - b)}{2b^2 + c^2 + a^2} \geq 0,
\]
\[
\sum (c + a + 2b)(a - b) \left( \frac{c + a + 2b}{2b^2 + c^2 + a^2} - \frac{b + c + 2a}{2a^2 + b^2 + c^2} \right) \geq 0,
\]
\[
\sum (3ab + bc + ca - c^2)(2c^2 + a^2 + b^2)(a - b)^2 \geq 0.
\]
Since \(3ab + bc + ca - c^2 \geq c(a + b - c)\), it suffices to show that
\[
\sum c(a + b - c)(2c^2 + a^2 + b^2)(a - b)^2 \geq 0.
\]
Assume that \(a \geq b \geq c\). It is enough to prove that
\[
(b + c - a)(2a^2 + b^2 + c^2)(b - c) + b(c + a - b)(2b^2 + c^2 + a^2)(c - a) \geq 0;
\]
that is,
\[
b(c + a - b)(2b^2 + c^2 + a^2)(a - c)^2 \geq a(a - b - c)(2a^2 + b^2 + c^2)(b - c)^2.
\]
Since \(c + a - b \geq a - b - c\), it suffices to prove that
\[
b(2b^2 + c^2 + a^2)(a - c)^2 \geq a(2a^2 + b^2 + c^2)(b - c)^2.
\]
We can obtain this inequality by multiplying the inequalities
\[
b^2(a - c)^2 \geq a^2(b - c)^2
\]
and
\[
a(2b^2 + c^2 + a^2) \geq b(2a^2 + b^2 + c^2).
\]
The last inequality is equivalent to
\[
(a - b)((a - b)^2 + ab + c^2) \geq 0.
\]

\[\square\]

**P 1.107.** If \(a, b, c\) are nonnegative real numbers, then
\[
\frac{(b + c)^2}{4a^2 + b^2 + c^2} + \frac{(c + a)^2}{4b^2 + c^2 + a^2} + \frac{(a + b)^2}{4c^2 + a^2 + b^2} \geq 2.
\]
*(Vasile Cîrtoaje, 2005)*
Solution. By the Cauchy-Schwarz inequality, we have

\[
\sum \frac{(b + c)^2}{4a^2 + b^2 + c^2} \geq \frac{[\sum (b + c)^2]^2}{\sum (b + c)^2(4a^2 + b^2 + c^2)}
\]

\[
= \frac{2[\sum a^4 + 3 \sum a^2b^2 + 4abc \sum a + 2 \sum ab(a^2 + b^2)]}{\sum a^4 + 5 \sum a^2b^2 + 4abc \sum a + \sum ab(a^2 + b^2)} \geq 2,
\]

since

\[
\sum ab(a^2 + b^2) \geq 2 \sum a^2b^2.
\]

The equality holds for \(a = b = c\), and for \(b = c = 0\) (or any cyclic permutation).

\(\square\)

P 1.108. If \(a, b, c\) are positive real numbers, then

\[(a)\quad \sum \frac{1}{11a^2 + 2b^2 + 2c^2} \leq \frac{3}{5(ab + bc + ca)};\]

\[(b)\quad \sum \frac{1}{4a^2 + b^2 + c^2} \leq \frac{1}{2(a^2 + b^2 + c^2)} + \frac{1}{ab + bc + ca}.\]

(Vasile Cîrtoaje, 2008)

Solution. We will prove that

\[
\sum \frac{k + 2}{ka^2 + b^2 + c^2} \leq \frac{11 - 2k}{a^2 + b^2 + c^2} + \frac{2(k - 1)}{ab + bc + ca}
\]

for any \(k > 1\). Due to homogeneity, we may assume that \(a^2 + b^2 + c^2 = 3\). On this hypothesis, we need to show that

\[
\sum \frac{k + 2}{(k - 1)a^2 + 3} \leq \frac{11 - 2k}{3} + \frac{2(k - 1)}{ab + bc + ca}.
\]

Using the substitution \(m = 3/(k - 1), m > 0\), the inequality can be written as

\[
m(m + 1) \sum \frac{1}{a^2 + m} \leq 3m - 2 + \frac{6}{ab + bc + ca}.
\]

By the Cauchy-Schwarz inequality, we have

\[(a^2 + m)[m + (m + 1 - a)^2] \geq [a \sqrt{m} + \sqrt{m}(m + 1 - a)]^2 = m(m + 1)^2,
\]

and hence

\[
\frac{m(m + 1)}{a^2 + m} \leq \frac{a^2 - 1}{m + 1} + m + 2 - 2a,
\]
\[ m(m+1) \sum \frac{1}{a^2 + m} \leq 3(m + 2) - 2 \sum a. \]

Thus, it suffices to show that
\[ 3(m + 2) - 2 \sum a \leq 3m - 2 + \frac{6}{ab + bc + ca}; \]
that is,
\[ (4 - a - b - c)(ab + bc + ca) \leq 3. \]

Let \( p = a + b + c \). Since
\[ 2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = p^2 - 3, \]
we get
\[ 6 - 2(4 - a - b - c)(ab + bc + ca) = 6 - (4 - p)(p^2 - 3) = (p - 3)^2(p + 2) \geq 0. \]

This completes the proof. The equality holds for \( a = b = c \).

\[ \square \]

**P 1.109.** If \( a, b, c \) are nonnegative real numbers such that \( ab + bc + ca = 3 \), then
\[ \frac{\sqrt{a}}{b + c} + \frac{\sqrt{b}}{c + a} + \frac{\sqrt{c}}{a + b} \geq \frac{3}{2}. \]

*(Vasile Cîrtoaje, 2006)*

**Solution.** By the Cauchy-Schwarz inequality, we have
\[ \sum \frac{\sqrt{a}}{b + c} \geq \left( \sum \frac{a^{3/4}}{b + c} \right)^2 \geq \frac{1}{6} \left( \sum a^{3/4} \right)^2. \]

Thus, it suffices to show that
\[ a^{3/4} + b^{3/4} + c^{3/4} \geq 3, \]
which follows immediately by Remark 1 from the proof of the inequality in P 2.33 in Volume 1. The equality occurs for \( a = b = c = 1 \).

**Remark.** Analogously, according to Remark 2 from the proof of P 2.33 in Volume 1, we can prove that
\[ \frac{a^k}{b + c} + \frac{b^k}{c + a} + \frac{c^k}{a + b} \geq \frac{3}{2} \]
for all \( k \geq 3 - \frac{4 \ln 2}{\ln 3} \approx 0.476 \). For \( k = 3 - \frac{4 \ln 2}{\ln 3} \), the equality occurs for \( a = b = c = 1 \), and also for \( a = 0 \) and \( b = c = \sqrt{3} \) (or any cyclic permutation).

\[ \square \]
P 1.110. If \(a, b, c\) are nonnegative real numbers such that \(ab + bc + ca \geq 3\), then

\[
\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \geq \frac{1}{1+b+c} + \frac{1}{1+c+a} + \frac{1}{1+a+b}.
\]

(Vasile Cîrtoaje, 2014)

Solution. Denote

\[E(a, b, c) = \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} - \frac{1}{1+b+c} - \frac{1}{1+c+a} - \frac{1}{1+a+b}.\]

Consider first the case \(ab + bc + ca = 3\). We will show that

\[
\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \geq \frac{1}{1+b+c} + \frac{1}{1+c+a} + \frac{1}{1+a+b}.
\]

By direct calculation, we can show that the left inequality is equivalent to \(abc \leq 1\). Indeed, applying the AM-GM inequality, we get

\[3 = ab + bc + ca \geq 3\sqrt{abc}.\]

Also, the right inequality is equivalent to

\[a + b + c \geq 2 + abc.\]

Since \(abc \leq 1\), it suffices to show that

\[a + b + c \geq 3.\]

Indeed,

\[(a + b + c)^2 \geq 3(ab + bc + ca) = 9.\]

Consider further that \(ab + bc + ca > 3\). Without loss of generality, assume that \(a \geq b \geq c\), \(a > 1\). For \(c \geq 1\), that is, \(a \geq b \geq c \geq 1\), the desired inequality follows by summing the obvious inequalities

\[
\frac{1}{2+a} \geq \frac{1}{1+c+a},
\]

\[
\frac{1}{2+b} \geq \frac{1}{1+a+b},
\]

\[
\frac{1}{2+c} \geq \frac{1}{1+b+c}.
\]

Therefore, assume now that \(c < 1\). Consider the cases \(b + c \geq 2\) and \(b + c < 2\).

Case 1: \(b + c \geq 2\), \(a > 1\), \(c < 1\). We will show that

\[E(a, b, c) \geq E(1, b, c) \geq 0.\]
We have
\[
E(a, b, c) - E(1, b, c) = \left( \frac{1}{2 + a} - \frac{1}{3} \right) + \left( \frac{1}{2 + b} - \frac{1}{1 + a + b} \right) + \left( \frac{1}{2 + c} - \frac{1}{1 + c + a} \right)
\]
\[
= (a - 1) \left[ \frac{-1}{3(2 + a)} + \frac{1}{(2 + b)(1 + a + b)} + \frac{1}{(2 + c)(1 + c + a)} \right]
\]
\[
> (a - 1) \left[ \frac{-1}{3(2 + a)} + \frac{1}{(2 + c)(1 + c + a)} \right]
\]
\[
= \frac{(a - 1)(1 - c)(4 + c + a)}{3(2 + a)(2 + c)(1 + c + a)} > 0
\]
and
\[
E(1, b, c) = \frac{b + c - 2}{3(1 + b + c)} \geq 0.
\]

**Case 2:** \(b + c < 2, a > 1, c < 1\). From \(b + c < 2\), it follows that
\[
bc \leq \left( \frac{b + c}{2} \right)^2 < 1.
\]
For fixed \(b\) and \(c\), define the function
\[
f(x) = E(x, b, c).
\]
Since
\[
f'(x) = \frac{-1}{(2 + x)^2} + \frac{1}{(1 + c + x)^2} + \frac{1}{(1 + x + b)^2} > \frac{-1}{(2 + x)^2} + \frac{1}{(1 + c + x)^2}
\]
\[
= \frac{(1 - c)(3 + 2x + c)}{(2 + x)^2(1 + c + x)^2} > 0,
\]
f(x) is strictly increasing for \(x \geq 0\). Since
\[
a > \frac{3 - bc}{b + c},
\]
we have \(f(a) > f\left( \frac{3 - bc}{b + c} \right)\). Therefore, it suffices to prove that \(f\left( \frac{3 - bc}{b + c} \right) \geq 0\), which is equivalent to \(E(a, b, c) \geq 0\) for \(a = \frac{3 - bc}{b + c}\), that is, \(ab + bc + ca = 3\). But this was proved in the first part of the proof. So, the proof is completed. The equality occurs for \(a = b = c = 1\). □
P 1.111. If \(a, b, c\) are the lengths of the sides of a triangle, then

\[
\begin{align*}
(a) & \quad \frac{a^2 - bc}{3a^2 + b^2 + c^2} + \frac{b^2 - ca}{3b^2 + c^2 + a^2} + \frac{c^2 - ab}{3c^2 + a^2 + b^2} \leq 0; \\
(b) & \quad \frac{a^4 - b^2c^2}{3a^4 + b^4 + c^4} + \frac{b^4 - c^2a^2}{3b^4 + c^4 + a^4} + \frac{c^4 - a^2b^2}{3c^4 + a^4 + b^4} \leq 0.
\end{align*}
\]

_Nguyen Anh Tuan and Vasile Cirtoaje, 2006_

**Solution.** (a) Apply the SOS method. We have

\[
2 \sum \frac{a^2 - bc}{3a^2 + b^2 + c^2} = \sum \frac{(a - b)(a + c) + (a - c)(a + b)}{3a^2 + b^2 + c^2} \\
\hspace*{1cm} = \sum \frac{(a - b)(a + c)}{3a^2 + b^2 + c^2} + \sum \frac{(b - a)(b + c)}{3b^2 + c^2 + a^2} \\
\hspace*{1cm} = \sum (a - b) \left( \frac{a + c}{3a^2 + b^2 + c^2} - \frac{b + c}{3b^2 + c^2 + a^2} \right) \\
\hspace*{1cm} = (a^2 + b^2 + c^2 - 2ab - 2bc - 2ca) \sum \frac{(a - b)^2}{(3a^2 + b^2 + c^2)(3b^2 + c^2 + a^2)}.
\]

Since

\[a^2 + b^2 + c^2 - 2ab - 2bc - 2ca = a(a - b - c) + b(b - c - a) + c(c - a - b) \leq 0,\]

the conclusion follows. The equality holds for an equilateral triangle, and for a degenerate triangle with \(a = 0\) and \(b = c\) (or any cyclic permutation).

(b) Using the same way as above, we get

\[
2 \sum \frac{a^4 - b^2c^2}{3a^4 + b^4 + c^4} = A \sum \frac{(a^2 - b^2)^2}{(3a^2 + b^2 + c^2)(3b^2 + c^2 + a^2)},
\]

where

\[
A = a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 \\
\hspace*{1cm} = -(a + b + c)(a + b - c)(b + c - a)(c + a - b) \leq 0.
\]

The equality holds for an equilateral triangle, and for a degenerate triangle with \(a = b+c\) (or any cyclic permutation). \(\Box\)
P 1.112. If \(a, b, c\) are the lengths of the sides of a triangle, then

\[
\frac{bc}{4a^2 + b^2 + c^2} + \frac{ca}{4b^2 + c^2 + a^2} + \frac{ab}{4c^2 + a^2 + b^2} \geq \frac{1}{2}.
\]

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2010)

**Solution.** We apply the SOS method. Write the inequality as

\[
\sum \left( \frac{2bc}{4a^2 + b^2 + c^2} - \sum \frac{b^2c^2}{a^2b^2 + b^2c^2 + c^2a^2} \right) \geq 0,
\]

\[
\sum \frac{bc(2a^2 - bc)(b - c)^2}{4a^2 + b^2 + c^2} \geq 0.
\]

Without loss of generality, assume that \(a \geq b \geq c\). Then, it suffices to prove that

\[
\frac{c(2b^2 - ca)(c - a)^2}{4b^2 + c^2 + a^2} + \frac{b(2c^2 - ab)(a - b)^2}{4c^2 + a^2 + b^2} \geq 0.
\]

Since

\[
2b^2 - ca \geq c(b + c) - ca = c(b + c - a) \geq 0
\]

and

\[
(2b^2 - ca) + (2c^2 - ab) = 2(b^2 + c^2) - a(b + c) \geq (b + c)^2 - a(b + c)
\]

\[
= (b + c)(b + c - a) \geq 0,
\]

it is enough to show that

\[
\frac{c(a - c)^2}{4b^2 + c^2 + a^2} \geq \frac{b(a - b)^2}{4c^2 + a^2 + b^2}.
\]

This follows by multiplying the inequalities

\[
c^2(a - c)^2 \geq b^2(a - b)^2
\]

and

\[
\frac{b}{4b^2 + c^2 + a^2} \geq \frac{c}{4c^2 + a^2 + b^2}.
\]

These inequalities are true, since

\[
c(a - c) - b(a - b) = (b - c)(b + c - a) \geq 0,
\]

\[
b(4c^2 + a^2 + b^2) - c(4b^2 + c^2 + a^2) = (b - c)[(b - c)^2 + a^2 - bc] \geq 0.
\]

The equality occurs for an equilateral triangle, and for a degenerate triangle with \(a = 0\) and \(b = c\) (or any cyclic permutation). \(\square\)
P 1.113. If \( a, b, c \) are the lengths of the sides of a triangle, then

\[
\frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{1}{a^2 + b^2} \leq \frac{9}{2(ab + bc + ca)}.
\]

(Vo Quoc Ba Can, 2008)

**Solution.** Apply the SOS method. Write the inequality as

\[
\sum \left[ \frac{3}{2} - \frac{ab + bc + ca}{b^2 + c^2} \right] \geq 0,
\]

\[
\sum \frac{3(b^2 + c^2) - 2(ab + bc + ca)}{b^2 + c^2} \geq 0,
\]

\[
\sum \frac{3b(b - a) + 3c(c - a) + c(a - b) + b(a - c)}{b^2 + c^2} \geq 0,
\]

\[
\sum \frac{(a - b)(c - 3b) + (a - c)(b - 3c)}{b^2 + c^2} \geq 0,
\]

\[
\sum \frac{(a - b)(c - 3b) + (a - c)(b - 3c)}{b^2 + c^2} + \sum \frac{b - a)(c - 3a)}{c^2 + a^2} \geq 0,
\]

\[
\sum (a^2 + b^2)(a - b)^2(c + c^2 - 3ab) \geq 0.
\]

Without loss of generality, assume that \( a \geq b \geq c \). Since \( ab + ac + 3a^2 - 3bc > 0 \), it suffices to prove that

\[
(a^2 + b^2)(a - b)^2(ca + cb + 3c^2 - 3ab) + (a^2 + c^2)(a - c)^2(ab + bc + 3b^2 - 3ac) \geq 0,
\]

or, equivalently,

\[
(a^2 + c^2)(a - c)^2(ab + bc + 3b^2 - 3ac) \geq (a^2 + b^2)(a - b)^2(3ab - 3c^2 - ca - cb).
\]

Since

\[
ab + bc + 3b^2 - 3ac = a \left( \frac{bc + 3b^2}{a} + b - 3c \right) \geq a \left( \frac{bc + 3b^2}{b + c} + b - 3c \right)
\]

\[
= \frac{a(b - c)(4b + 3c)}{b + c} \geq 0
\]

and

\[
(ab + bc + 3b^2 - 3ac) - (3ab - 3c^2 - ca - cb) = 3(b^2 + c^2) + 2bc - 2a(b + c)
\]

\[
\geq 3(b^2 + c^2) + 2bc - 2(b + c)^2 = (b - c)^2 \geq 0,
\]
it suffices to show that
\[(a^2 + c^2)(a - c)^2 \geq (a^2 + b^2)(a - b)^2.\]

This is true, since it is equivalent to \((b - c)A \geq 0\), where
\[
A = 2a^3 - 2a^2(b + c) + 2a(b^2 + bc + c^2) - (b + c)(b^2 + c^2) \\
= 2a\left(a - \frac{b + c}{2}\right)^2 + \frac{a(3b^2 + 2bc + 3c^2)}{2} - (b + c)(b^2 + c^2) \\
\geq \frac{b(3b^2 + 2bc + 3c^2)}{2} - (b + c)(b^2 + c^2) \\
= \frac{(b - c)(b^2 + bc + 2c^2)}{2} \geq 0.
\]
The equality occurs for an equilateral triangle, and for a degenerate triangle with \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.114.** If \(a, b, c\) are the lengths of the sides of a triangle, then

(a) \[\frac{a + b}{a - b} + \frac{b + c}{b - c} + \frac{c + a}{c - a} > 5;\]

(b) \[\frac{a^2 + b^2}{a^2 - b^2} + \frac{b^2 + c^2}{b^2 - c^2} + \frac{c^2 + a^2}{c^2 - a^2} \geq 3.\]

*(Vasile Cîrtoaje, 2003)*

**Solution.** Since the inequalities are symmetric, we consider that \(a > b > c\).

(a) Let \(x = a - c\) and \(y = b - c\). From \(a > b > c\) and \(a \leq b + c\), it follows that \(x > y > 0\) and \(c \geq x - y\). We have

\[
\frac{a + b}{a - b} + \frac{b + c}{b - c} + \frac{c + a}{c - a} = \frac{2c + x + y}{x - y} + \frac{2c + y}{y} - \frac{2c + x}{x} \\
= 2c\left(\frac{1}{x - y} + \frac{1}{y} - \frac{1}{x}\right) + \frac{x + y}{x - y} > \frac{2c}{y} + \frac{x + y}{x - y} \\
\geq \frac{2(x - y)}{y} + \frac{x + y}{x - y} = 2\left(\frac{x - y}{y} + \frac{y}{x - y}\right) + 1 \geq 5.
\]
(b) We will show that
\[
\frac{a^2 + b^2}{a^2 - b^2} + \frac{b^2 + c^2}{b^2 - c^2} + \frac{c^2 + a^2}{c^2 - a^2} \geq 3;
\]
that is,
\[
\frac{b^2}{a^2 - b^2} + \frac{c^2}{b^2 - c^2} \geq \frac{a^2}{a^2 - c^2}.
\]
Since
\[
\frac{a^2}{a^2 - c^2} \leq \frac{(b + c)^2}{a^2 - c^2},
\]
it suffices to prove that
\[
\frac{b^2}{a^2 - b^2} + \frac{c^2}{b^2 - c^2} \geq \frac{(b + c)^2}{a^2 - c^2}.
\]
This is equivalent to each of the following inequalities:
\[
b^2 \left( \frac{1}{a^2 - b^2} - \frac{1}{a^2 - c^2} \right) + c^2 \left( \frac{1}{b^2 - c^2} - \frac{1}{a^2 - c^2} \right) \geq \frac{2bc}{a^2 - c^2},
\]
\[
b^2 \frac{(b^2 - c^2)}{a^2 - b^2} + c^2 \frac{(a^2 - b^2)}{b^2 - c^2} \geq 2bc,
\]
\[
[b(b^2 - c^2) - c(a^2 - b^2)]^2 \geq 0.
\]
This completes the proof. If \( a > b > c \), then the equality holds for a degenerate triangle with \( a = b + c \) and \( b/c = x_1 \), where \( x_1 \approx 1.5321 \) is the positive root of the equation \( x^3 - 3x - 1 = 0 \).

\[\square\]

**P 1.115.** If \( a, b, c \) are the lengths of the sides of a triangle, then
\[
\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} + 3 \geq 6 \left( \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \right).
\]

**Solution.** We apply the SOS method. Write the inequality as
\[
\sum \frac{b + c}{a} - 6 \geq 3 \left( \sum \frac{2a}{b + c} - 3 \right).
\]
Since
\[
\sum \frac{b + c}{a} - 6 = \sum \left( \frac{b}{c} + \frac{c}{b} \right) - 6 = \sum \frac{(b - c)^2}{bc}
\]
and
\[
\sum \frac{2a}{b + c} - 3 = \sum \frac{2a - b - c}{b + c} = \sum \frac{a - b}{b + c} + \sum \frac{a - c}{b + c}
\]
we can rewrite the inequality as
\[ \sum a(b + c)S_a(b - c)^2 \geq 0, \]
where
\[ S_a = a(a + b + c) - 2bc. \]
Without loss of generality, assume that \( a \geq b \geq c \). Since \( S_a > 0 \),
\[ S_b = b(a + b + c) - 2ca = (b - c)(a + b + c) + c(b + c - a) \geq 0 \]
and
\[ \sum a(b + c)S_a(b - c)^2 \geq b(c + a)S_b(c - a)^2 + c(a + b)S_c(a - b)^2 \geq (a - b)^2[b(c + a)S_b + c(a + b)S_c], \]
it suffices to prove that
\[ b(c + a)S_b + c(a + b)S_c \geq 0. \]
This is equivalent to each of the following inequalities
\[ (a + b + c)[a(b^2 + c^2) + bc(b + c)] \geq 2abc(2a + b + c), \]
\[ a(a + b + c)(b - c)^2 + (a + b + c)[2abc + bc(b + c)] \geq 2abc(2a + b + c), \]
\[ a(a + b + c)(b - c)^2 + bc(2a + b + c)(b + c - a) \geq 0. \]
Since the last inequality is true, the proof is completed. The equality occurs for an equilateral triangle, and for a degenerate triangle with \( a/2 = b = c \) (or any cyclic permutation).

\[ \square \]

**P 1.116.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[ \sum \frac{3a(b + c) - 2bc}{(b + c)(2a + b + c)} \geq \frac{3}{2}. \]
(\textit{Vasile Cirtoaje, 2009})

**Solution.** Use the SOS method. Write the inequality as follows
\[ \sum \left[ \frac{3a(b + c) - 2bc}{(b + c)(2a + b + c)} - \frac{1}{2} \right] \geq 0, \]
the conclusion follows. The equality holds for \(a = b\), or \(b = c\), or \(c = a\).

\(\square\)

**P 1.17.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

\[
\sum \frac{a(b + c) - 2bc}{(b + c)(3a + b + c)} \geq 0.
\]

\((\text{Vasile Cirtoaje}, 2009)\)

**Solution.** We apply the SOS method. Since

\[
\sum \frac{a(b + c) - 2bc}{(b + c)(3a + b + c)} = \sum \frac{b(a - c) + c(a - b)}{(b + c)(3a + b + c)}
\]

\[
= \sum \frac{c(b - a)}{(c + a)(3b + c + a)} + \sum \frac{c(a - b)}{(b + c)(3a + b + c)}
\]

\[
= \sum \frac{c(a + b - c)(a - b)^2}{(b + c)(c + a)(3a + b + c)(3b + c + a)},
\]

the inequality is equivalent to

\[
\sum c(a + b)(3c + a + b)(a + b - c)(a - b)^2 \geq 0.
\]

Without loss of generality, assume that \(a \geq b \geq c\). Since \(a + b - c \geq 0\), it suffices to show that

\[
b(c + a)(3b + c + a)(c + a - b)(a - c)^2 \geq a(b + c)(3a + b + c)(a - b - c)(b - c)^2.
\]
This is true since
\[
\begin{align*}
  c + a - b &\geq a - b - c, \\
  b^2(a - c)^2 &\geq a^2(b - c)^2, \\
  c + a &\geq b + c, \\
  a(b + c + a) &\geq b(3a + b + c).
\end{align*}
\]
The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

**P 1.118.** Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 \geq 3\). Prove that
\[
\frac{a^5 - a^2}{a^5 + b^2 + c^2} + \frac{b^5 - b^2}{b^5 + c^2 + a^2} + \frac{c^5 - c^2}{c^5 + a^2 + b^2} \geq 0.
\]

*(Vasile Cîrtoaje, 2005)*

**Solution.** The inequality is equivalent to
\[
\frac{1}{a^5 + b^2 + c^2} + \frac{1}{b^5 + c^2 + a^2} + \frac{1}{c^5 + a^2 + b^2} \leq \frac{3}{a^2 + b^2 + c^2}.
\]
Setting \(a = tx, b = ty\) and \(c = tz\), where \(t > 0\) and \(x, y, z > 0\) such that \(x^2 + y^2 + z^2 = 3\), the condition \(a^2 + b^2 + c^2 \geq 3\) implies \(t \geq 1\), and the inequality becomes
\[
\frac{1}{t^3x^5 + y^2 + z^2} + \frac{1}{t^3y^5 + z^2 + x^2} + \frac{1}{t^3z^5 + x^2 + y^2} \leq 1.
\]
We see that it suffices to prove this inequality for \(t = 1\), when it becomes
\[
\frac{1}{x^5 - x^2 + 3} + \frac{1}{y^5 - y^2 + 3} + \frac{1}{z^5 - z^2 + 3} \leq 1.
\]
Without loss of generality, assume that \(x \geq y \geq z\). There are two cases to consider.

**Case 1:** \(z \leq y \leq x \leq \sqrt{2}\). The desired inequality follows by adding the inequalities
\[
\frac{1}{x^5 - x^2 + 3} \leq \frac{3 - x^2}{6}, \quad \frac{1}{y^5 - y^2 + 3} \leq \frac{3 - y^2}{6}, \quad \frac{1}{z^5 - z^2 + 3} \leq \frac{3 - z^2}{6}.
\]
We have
\[
\frac{1}{x^5 - x^2 + 3} - \frac{3 - x^2}{6} = \frac{(x - 1)^2(x^5 + 2x^4 - 3x^2 - 6x - 3)}{6(x^5 - x^2 + 3)} \leq 0,
\]
since
\[
x^5 + 2x^4 - 3x^2 - 6x - 3 = x^2(x^3 + 2x^2 - 3 - \frac{6}{x} + \frac{3}{x^2})
\]
\[
\leq x^2(2\sqrt{2} + 4 - 3 - 3\sqrt{2} - \frac{3}{2}) = -x^2(\sqrt{2} + \frac{1}{2}) < 0.
\]

Case 2: \(x > \sqrt{2}\). From \(x^2 + y^2 + z^2 = 3\), it follows that \(y^2 + z^2 < 1\). Since
\[
\frac{1}{x^5 - x^2 + 3} < \frac{1}{(2\sqrt{2} - 1)x^2 + 3} < \frac{1}{2(2\sqrt{2} - 1) + 3} < \frac{1}{6}
\]
and
\[
\frac{1}{y^5 - y^2 + 3} + \frac{1}{z^5 - z^2 + 3} < \frac{1}{3 - y^2 + \frac{1}{3 - z^2}},
\]
it suffices to prove that
\[
\frac{1}{3 - y^2} + \frac{1}{3 - z^2} \leq \frac{5}{6}.
\]
Indeed, we have
\[
\frac{1}{3 - y^2} + \frac{1}{3 - z^2} - \frac{5}{6} = \frac{9(y^2 + z^2 - 1) - 5y^2z^2}{6(3 - y^2)(3 - z^2)} < 0,
\]
which completes the proof. The equality occurs for \(a = b = c = 1\).

Remark. Since \(abc \geq 1\) involves \(a^2 + b^2 + c^2 \geq 3\sqrt[3]{a^2b^2c^2} \geq 3\), the original inequality is also true for \(abc \geq 1\), which is a problem from IMO-2005 (by Hojoo Lee). A proof of this inequality is the following:
\[
\sum \frac{a^5 - a^2}{a^5 + b^2 + c^2} \geq \sum \frac{a^3 - 1}{a(a^2 + b^2 + c^2)}
\]
\[
= \frac{1}{a^2 + b^2 + c^2} \sum (a^2 - \frac{1}{a}) \geq \frac{1}{a^2 + b^2 + c^2} \sum (a^2 - bc)
\]
\[
= \frac{1}{2(a^2 + b^2 + c^2)} \sum (a - b)^2 \geq 0.
\]

P 1.119. Let \(a, b, c\) be positive real numbers such that \(a^2 + b^2 + c^2 = a^3 + b^3 + c^3\). Prove that
\[
\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b} \geq \frac{3}{2}.
\]

(Pham Huu Duc, 2008)
**First Solution.** By the Cauchy-Schwarz inequality, we have
\[
\sum a^2 \geq \frac{(\sum a^3)^2}{\sum a^4(b + c)} = \frac{(\sum a^3)(\sum a^2)}{(\sum a^2)(\sum ab) - abc \sum a^2}.
\]
Therefore, it is enough to show that
\[
2(\sum a^3)(\sum a^2) + 3abc \sum a^2 \geq 3(\sum a^3)(\sum ab).
\]
Write this inequality as follows
\[
3(\sum a^3)(\sum a^2 - \sum ab) - (\sum a^3 - 3abc) \sum a^2 \geq 0,
\]
\[
3(\sum a^3)(\sum a^2 - \sum ab) - (\sum a)(\sum a^2 - \sum ab) \sum a^2 \geq 0,
\]
\[
(\sum a^2 - \sum ab)[3 \sum a^3 - (\sum a)(\sum a^2)] \geq 0.
\]
The last inequality is true, since
\[
2(\sum a^2 - \sum ab) = \sum (a - b)^2 \geq 0
\]
and
\[
3 \sum a^3 - (\sum a)(\sum a^2) = \sum (a^3 + b^3) - \sum ab(a + b) = \sum (a + b)(a - b)^2 \geq 0.
\]
The equality occurs for \(a = b = c = 1\).

**Second Solution.** Write the inequality in the homogeneous form \(A \geq B\), where
\[
A = 2 \sum \frac{a^2}{b + c} - \sum a, \quad B = \frac{3(a^2 + b^3 + c^3)}{a^2 + b^2 + c^2} - \sum a.
\]
Since
\[
A = \sum \frac{a(a - b) + a(a - c)}{b + c} = \sum \frac{a(a - b)}{b + c} + \sum \frac{b(b - a)}{c + a} = (a + b + c) \sum \frac{(a - b)^2}{(b + c)(c + a)}
\]
and
\[
B = \frac{\sum(a^3 + b^3) - \sum ab(a + b)}{a^2 + b^2 + c^2} = \frac{\sum(a + b)(a - b)^2}{a^2 + b^2 + c^2},
\]
we can write the inequality as
\[
\sum \left[ \frac{a + b + c}{(b + c)(c + a)} - \frac{a + b}{a^2 + b^2 + c^2} \right] (a - b)^2 \geq 0,
\]
\[
(a^3 + b^3 + c^3 - 2abc) \sum \frac{(a - b)^2}{(b + c)(c + a)} \geq 0.
\]
Since \(a^3 + b^3 + c^3 \geq 3abc\), the conclusion follows.
P 1.120. If \(a, b, c \in [0, 1]\), then

\[
\begin{align*}
(a) & \quad \frac{a}{bc + 2} + \frac{b}{ca + 2} + \frac{c}{ab + 2} \leq 1; \\
(b) & \quad \frac{ab}{2bc + 1} + \frac{bc}{2ca + 1} + \frac{ca}{2ab + 1} \leq 1.
\end{align*}
\]

*Solution.* (a) It suffices to show that

\[
\frac{a}{abc + 2} + \frac{b}{abc + 2} + \frac{c}{abc + 2} \leq 1,
\]

which is equivalent to

\[abc + 2 \geq a + b + c.\]

We have

\[abc + 2 - a - b - c = (1 - b)(1 - c) + (1 - a)(1 - bc) \geq 0.\]

The equality holds for \(a = b = c = 1\), and for \(a = 0\) and \(b = c = 1\) (or any cyclic permutation).

(b) It suffices to prove that

\[
\frac{ab}{2abc + 1} + \frac{bc}{2abc + 1} + \frac{ca}{2abc + 1} \leq 1;
\]

that is,

\[2abc + 1 \geq ab + bc + ca.\]

Since

\[a + b + c - (ab + bc + ca) = a(1 - b) + b(1 - c) + c(1 - a) \geq 0,\]

we have

\[2abc + 1 - ab - bc - ca \geq 2abc + 1 - a - b - c = (1 - b)(1 - c) + (1 - a)(1 - bc) \geq 0.\]

The equality holds for \(a = b = c = 1\), and for \(a = 0\) and \(b = c = 1\) (or any cyclic permutation).

\[\square\]

P 1.121. Let \(a, b, c\) be positive real numbers such that \(a + b + c = 2\). Prove that

\[
5(1 - ab - bc - ca)\left(\frac{1}{1 - ab} + \frac{1}{1 - bc} + \frac{1}{1 - ca}\right) + 9 \geq 0.
\]

*Solution.* (Vasile Cirtoaje, 2011)
**Solution.** Write the inequality as 

\[
24 - \frac{5a(b + c)}{1 - bc} - \frac{5b(c + a)}{1 - ca} - \frac{5c(a + b)}{1 - ab} \geq 0.
\]

Since 

\[
4(1 - bc) \geq 4 - (b + c)^2 = (a + b + c)^2 - (b + c)^2 = a(a + 2b + 2c),
\]

it suffices to show that

\[
6 - 5 \left( \frac{b + c}{a + 2b + 2c} - \frac{c + a}{b + 2c + 2a} - \frac{a + b}{c + 2a + 2b} \right) \geq 0,
\]

which is equivalent to

\[
\sum 5 \left( 1 - \frac{b + c}{a + 2b + 2c} \right) \geq 9,
\]

\[
5(a + b + c) \sum \frac{1}{a + 2b + 2c} \geq 9,
\]

\[
\left[ \sum (a + 2b + 2c) \right] \left[ \sum \frac{1}{a + 2b + 2c} \right] \geq 9.
\]

The last inequality follows immediately from the AM-HM inequality. The equality holds for \(a = b = c = 2/3\).

□

**P 1.122.** Let \(a, b, c\) be nonnegative real numbers such that \(a + b + c = 2\). Prove that

\[
\frac{2 - a^2}{2 - bc} + \frac{2 - b^2}{2 - ca} + \frac{2 - c^2}{2 - ab} \leq 3.
\]

*(Vasile Cîrtoaje, 2011)*

**First Solution.** Write the inequality as follows

\[
\sum \left( 1 - \frac{2 - a^2}{2 - bc} \right) \geq 0,
\]

\[
\sum \frac{a^2 - bc}{2 - bc} \geq 0,
\]

\[
\sum (a^2 - bc)(2 - ca)(2 - ab) \geq 0,
\]

\[
\sum (a^2 - bc)[4 - 2a(b + c) + a^2bc] \geq 0,
\]

\[
4 \sum (a^2 - bc) - 2 \sum a(b + c)(a^2 - bc) + abc \sum a(a^2 - bc) \geq 0.
\]
By virtue of the AM-GM inequality,

\[ \sum a(a^2 - bc) = a^3 + b^3 + c^3 - 3abc \geq 0. \]

Then, it suffices to prove that

\[ 2 \sum (a^2 - bc) \geq \sum (a(b + c)(a^2 - bc)). \]

Indeed, we have

\[
\begin{align*}
\sum a(b + c)(a^2 - bc) &= \sum a^3(b + c) - abc \sum (b + c) \\
&= \sum a(b^3 + c^3) - abc \sum (b + c) = \sum a(b + c)(b - c)^2 \\
&\leq \sum \left[ \frac{a + (b + c)}{2} \right]^2 (b - c)^2 = (b - c)^2 = 2 \sum (a^2 - bc).
\end{align*}
\]

The equality holds for \( a = b = c = \frac{2}{3} \), and for \( a = 0 \) and \( b = c = 1 \) (or any cyclic permutation).

**Second Solution.** We apply the SOS method. Write the inequality as follows

\[
\begin{align*}
\sum &\frac{a^2 - bc}{2 - bc} \geq 0, \\
\sum &\frac{(a - b)(a + c) + (a - c)(a + b)}{2 - bc} \geq 0, \\
\sum &\frac{(a - b)(a + c)}{2 - bc} + \sum \frac{(b - a)(b + c)}{2 - ca} \geq 0, \\
\sum &\frac{(a - b)^2(2 - c(a + b) - c^2)}{(2 - bc)(2 - ca)} \geq 0, \\
\sum & (a - b)^2(2 - ab)(1 - c) \geq 0.
\end{align*}
\]

Assuming that \( a \geq b \geq c \), it suffices to prove that

\[ (b - c)^2(2 - bc)(1 - a) + (c - a)^2(2 - ca)(1 - b) \geq 0. \]

Since \( 2(1 - b) = a - b + c \geq 0 \) and \( (c - a)^2 \geq (b - c)^2 \), it suffices to show that

\[ (2 - bc)(1 - a) + (2 - ca)(1 - b) \geq 0. \]

We have

\[
\begin{align*}
(2 - bc)(1 - a) + (2 - ca)(1 - b) &= 4 - 2(a + b) - c(a + b) + 2abc \\
&\geq 4 - (a + b)(2 + c) \geq 4 - \left( \frac{(a + b) + (2 + c)}{2} \right)^2 = 0.
\end{align*}
\]
P 1.123. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that
\[
\frac{3 + 5a^2}{3 - bc} + \frac{3 + 5b^2}{3 - ca} + \frac{3 + 5c^2}{3 - ab} \geq 12.
\]
(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Write the inequality as follows
\[
\sum \left( \frac{3 + 5a^2}{3 - bc} - 4 \right) \geq 0,
\]
\[
\sum \frac{5a^2 + 4bc - 9}{3 - bc} \geq 0,
\]
\[
\sum \frac{4a^2 - b^2 - c^2 - 2ab + 2bc - 2ca}{3 - bc} \geq 0,
\]
\[
\sum \frac{(2a^2 - b^2 - c^2) + 2(a - b)(a - c)}{3 - bc} \geq 0,
\]
\[
\sum \frac{([a - b](a + b) + [a - b](a - c)) + [(a - c)(a + c) + (a - c)(a - b)]}{3 - bc} \geq 0,
\]
\[
\sum \frac{(a - b)(2a + b - c) + (a - c)(2a + c - b)}{3 - bc} \geq 0,
\]
\[
\sum \frac{(a - b)(2a + b - c) + (b - a)(2b + a - c)}{3 - ca} \geq 0,
\]
\[
\sum \frac{(a - b)^2[3 - 2c(a + b) + c^2]}{(3 - bc)(3 - ca)} \geq 0,
\]
\[
\sum \frac{(a - b)^2(c - 1)^2}{(3 - bc)(3 - ca)} \geq 0.
\]
The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

\(\square\)

P 1.124. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 2$. If
\[
-\frac{1}{7} \leq m \leq \frac{7}{8},
\]
then
\[
\frac{a^2 + m}{3 - 2bc} + \frac{b^2 + m}{3 - 2ca} + \frac{c^2 + m}{3 - 2ab} \geq \frac{3(4 + 9m)}{19}.
\]
(Vasile Cîrtoaje, 2010)
Solution. We apply the SOS method. Write the inequality as
\[
\sum \left( \frac{a^2 + m}{3 - 2bc} - \frac{4 + 9m}{19} \right) \geq 0,
\]
\[
\sum \frac{19a^2 + 2(4 + 9m)bc - 12 - 8m}{3 - 2bc} \geq 0.
\]
Since
\[
19a^2 + 2(4 + 9m)bc - 12 - 8m = 19a^2 + 2(4 + 9m)bc - (3 + 2m)(a + b + c)^2
\]
\[
= (16 - 2m)a^2 - (3 + 2m)(b^2 + c^2 + 2ab + 2ac) + 2(1 + 7m)bc
\]
\[
= (3 + 2m)(2a^2 - b^2 - c^2) + 2(5 - 3m)(a^2 + bc - ab - ac) + (4 - 10m)(ab + ac - 2bc)
\]
\[
= (3 + 2m)(a^2 - b^2) + (5 - 3m)(a - b)(a - c) + (4 - 10m)c(a - b)
\]
\[
+ (3 + 2m)(a^2 - c^2) + (5 - 3m)(a - c)(a - b) + (4 - 10m)b(a - c)
\]
\[
= (a - b)B + (a - c)C,
\]
where
\[
B = (8 - m)a + (3 + 2m)b - (1 + 7m)c,
\]
\[
C = (8 - m)a + (3 + 2m)c - (1 + 7m)b,
\]
the inequality can be written as
\[
B_1 + C_1 \geq 0,
\]
where
\[
B_1 = \sum \frac{(a - b)[(8 - m)a + (3 + 2m)b - (1 + 7m)c]}{3 - 2bc},
\]
\[
C_1 = \sum \frac{(b - a)[(8 - m)b + (3 + 2m)a - (1 + 7m)c]}{3 - 2ca}.
\]
We have
\[
B_1 + C_1 = \sum \frac{(a - b)^2E_c}{(3 - 2bc)(3 - 2ca)},
\]
where
\[
E_c = 3(5 - 3m) - 2(8 - m)c(a + b) + 2(1 + 7m)c^2
\]
\[
= 6(2m + 3)c^2 - 4(8 - m)c + 3(5 - 3m)
\]
\[
= 6(2m + 3) \left[ c - \frac{8 - m}{3(2m + 3)} \right]^2 + \frac{(1 + 7m)(7 - 8m)}{3(2m + 3)}.
\]
Since \( E_c \geq 0 \) for \(-1/7 \leq m \leq 7/8\), we get \( B_1 + C_1 \geq 0 \). Thus, the proof is completed. The equality holds for \( a = b = c = 2/3 \). When \( m = -1/7 \), the equality holds for
$a = b = c = 2/3$, and for $a = 0$ and $b = c = 1$ (or any cyclic permutation). When $m = 7/8$, the equality holds for $a = b = c = 2/3$, and for $a = 1$ and $b = c = 1/2$ (or any cyclic permutation).

**Remark.** The inequalities in P 1.123 and P 1.124 are particular cases ($k = 3$ and $k = 8/3$, respectively) of the following more general result:

- Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. For $0 < k \leq 3$ and $m_1 \leq m \leq m_2$, where

$$m_1 = \begin{cases} -\infty, & 0 < k \leq \frac{3}{2} \\ \frac{(3-k)(4-k)}{2(3-2k)}, & \frac{3}{2} < k \leq 3 \end{cases},$$

$$m_2 = \frac{36 - 4k - k^2 + 4(9-k)\sqrt{3(3-k)}}{72 + k},$$

then

$$\frac{a^2 + mbc}{9 - kbc} + \frac{b^2 + mca}{9 - kca} + \frac{c^2 + mab}{9 - kab} \geq \frac{3(1 + m)}{9 - k},$$

with equality for $a = b = c = 1$. When $m = m_1$ and $3/2 < k \leq 3$, the equality holds also for

$$a = 0, \quad b = c = \frac{3}{2}.$$  

When $m = m_2$, the equality holds also for

$$a = \frac{3k - 6 + 2\sqrt{3(3-k)}}{k}, \quad b = c = \frac{3 - \sqrt{3(3-k)}}{k}.$$  

\[\square\]

**P 1.125.** Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{47 - 7a^2}{1 + bc} + \frac{47 - 7b^2}{1 + ca} + \frac{47 - 7c^2}{1 + ab} \geq 60.$$

*(Vasile Cirtoaje, 2011)*

**Solution.** We apply the SOS method. Write the inequality as follows

$$\sum \left( \frac{47 - 7a^2}{1 + bc} - 20 \right) \geq 0,$$
\[ \sum \frac{27 - 7a^2 - 20bc}{1 + bc} \geq 0, \]
\[ \sum \frac{3(a + b + c)^2 - 7a^2 - 20bc}{1 + bc} \geq 0, \]
\[ \sum \frac{-3(2a^2 - b^2 - c^2) + 2(a - b)(a - c) + 8(ab - 2bc + ca)}{1 + bc} \geq 0, \]
\[ \sum \frac{-3(a - b)(a + b) + (a - b)(a - c) + 8c(a - b)}{1 + bc} + \sum \frac{-3(a - c)(a + c) + (a - c)(a - b) + 8b(a - c)}{1 + bc} \geq 0, \]
\[ \sum \frac{(a - b)(-2a - 3b + 7c)}{1 + bc} + \sum \frac{(a - c)(-2a - 3c + 7b)}{1 + bc} \geq 0, \]
\[ \sum \frac{(a - b)(-2a - 3b + 7c)}{1 + bc} + \sum \frac{(b - a)(-2b - 3a + 7c)}{1 + ca} \geq 0, \]
\[ \sum \frac{(a - b)^2[1 - 2c(a + b) + 7c^2]}{(1 + bc)(1 + ca)} \geq 0, \]
\[ \sum \frac{(a - b)^2(3c - 1)^2}{(1 + bc)(1 + ca)} \geq 0, \]

The equality holds for \( a = b = c = 1 \), and for \( a = 7/3 \) and \( b = c = 1/3 \) (or any cyclic permutation).

**Remark.** The inequality in P 1.125 is a particular cases \( k = 9 \) of the following more general result:

- Let \( a, b, c \) be nonnegative real numbers such that \( a + b + c = 3 \). For \( k > 0 \) and \( m \geq m_1 \), where

\[
m_1 = \begin{cases} 
  \frac{36 + 4k - k^2 + 4(9 + k)\sqrt{3(3 + k)}}{72 - k}, & k \neq 72 \\
  \frac{238}{5}, & k = 72 
\end{cases},
\]

then

\[ \frac{a^2 + mbc}{9 + kbc} + \frac{b^2 + mca}{9 + kca} + \frac{c^2 + mab}{9 + kab} \leq \frac{3(1 + m)}{9 + k}, \]

with equality for \( a = b = c = 1 \). When \( m = m_1 \), the equality holds also for

\[ a = \frac{3k + 6 - 2\sqrt{3(3 + k)}}{k}, \quad b = c = \frac{\sqrt{3(3 + k)} - 3}{k}. \]
Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{26 - 7a^2}{1 + bc} + \frac{26 - 7b^2}{1 + ca} + \frac{26 - 7c^2}{1 + ab} \leq \frac{57}{2}.$$  

(Vasile Cîrtoaje, 2011)

**Solution.** Use the SOS method. Write the inequality as follows

$$\sum \left( \frac{19}{2} - \frac{26 - 7a^2}{1 + bc} \right) \geq 0,$$

$$\sum \frac{14a^2 + 19bc - 33}{1 + bc} \geq 0,$$

$$\sum \frac{42a^2 + 57bc - 11(a + b + c)^2}{1 + bc} \geq 0,$$

$$\sum \frac{11(2a^2 - b^2 - c^2) + 9(a - b)(a - c) - 13(ab - 2bc + ca)}{1 + bc} \geq 0,$$

$$\sum \frac{22(a - b)(a + b) + 9(a - b)(a - c) - 26c(a - b) + 22(a - c)(a + c) + 9(a - c)(a - b) - 26b(a - c)}{1 + bc} \geq 0,$$

$$\sum \frac{(a - b)(31a + 22b - 35c)}{1 + bc} + \sum \frac{(a - c)(31a + 22c - 35b)}{1 + bc} \geq 0,$$

$$\sum \frac{(a - b)(31a + 22b - 35c)}{1 + bc} + \sum \frac{(b - a)(31b + 22a - 35c)}{1 + ca} \geq 0,$$

$$\sum \frac{(a - b)^2 (9 + 31c(a + b) - 35c^2)}{(1 + bc)(1 + ca)} \geq 0,$$

$$\sum (a - b)^2 (1 + ab)(1 + 11c)(3 - 2c) \geq 0.$$  

Assume that $a \geq b \geq c$. Since $3 - 2c > 0$, it suffices to show that

$$(b - c)^2 (1 + bc)(1 + 11a)(3 - 2a) + (c - a)^2 (1 + ab)(1 + 11b)(3 - 2b) \geq 0;$$

that is,

$$(a - c)^2 (1 + ab)(1 + 11b)(3 - 2b) \geq (b - c)^2 (1 + bc)(1 + 11a)(2a - 3).$$

Since $3 - 2b = a - b + c \geq 0$, we get this inequality by multiplying the inequalities

$$3 - 2b \geq 2a - 3,$$

$$a(1 + ab) \geq b(1 + bc),$$
\[ a(1 + 11b) \geq b(1 + 11a), \]
\[ b^2(a - c)^2 \geq a^2(b - c)^2. \]
The equality holds for \( a = b = c = 1 \), and for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

**Remark.** The inequalities in P 1.126 is a particular cases \((k = 9)\) of the following more general result:

- Let \( a, b, c \) be nonnegative real numbers such that \( a + b + c = 3 \). For \( k > 0 \) and \( m \leq m_2 \), where
  \[ m_2 = \frac{(3 + k)(4 + k)}{2(3 + 2k)}, \]
  then
  \[ \frac{a^2 + mbc}{9 + kbc} + \frac{b^2 + mca}{9 + kca} + \frac{c^2 + mab}{9 + kab} \geq \frac{3(1 + m)}{9 + k}, \]
with equality for \( a = b = c = 1 \). When \( m = m_2 \), the equality holds also for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

\[ \square \]

**P 1.127.** Let \( a, b, c \) be nonnegative real numbers, no all are zero. Prove that

\[ \sum \frac{5a(b + c) - 6bc}{a^2 + b^2 + c^2 + bc} \leq 3. \]

*(Vasile Cirtoaje, 2010)*

**First Solution.** Apply the SOS method. If two of \( a, b, c \) are zero, then the inequality is trivial. Consider further that \( a^2 + b^2 + c^2 = 1 \), \( a \geq b \geq c \), \( b > 0 \), and write the inequality as follows

\[ \sum \left[ 1 - \frac{5a(b + c) - 6bc}{a^2 + b^2 + c^2 + bc} \right] \geq 0, \]
\[ \sum \frac{a^2 + b^2 + c^2 - 5a(b + c) + 7bc}{a^2 + b^2 + c^2 + bc} \geq 0, \]
\[ \sum \frac{(7b + 2c - a)(c - a) - (7c + 2b - a)(a - b)}{1 + bc} \geq 0, \]
\[ \sum \frac{(7c + 2a - b)(a - b)}{1 + ca} - \sum \frac{(7c + 2b - a)(a - b)}{1 + bc} \geq 0, \]
\[ \sum (a - b)^2(1 + ab)(3 + ac + bc - 7c^2) \geq 0. \]
Symmetric Rational Inequalities

Since
\[3 + ac + bc - 7c^2 = 3a^2 + 3b^2 + ac + bc - 4c^2 > 0,\]
it suffices to prove that
\[(1 + bc)(3 + ab + ac - 7a^2)(b - c)^2 + (1 + ac)(3 + ab + bc - 7b^2)(a - c)^2 \geq 0.\]

Since
\[3 + ab + ac - 7b^2 = 3(a^2 - b^2) + 3c^2 + b(a - b) + bc \geq 0\]
and \(1 + ac \geq 1 + bc\), it is enough to show that
\[(3 + ab + ac - 7a^2)(b - c)^2 + (3 + ab + bc - 7b^2)(a - c)^2 \geq 0.\]

From \(b(a - c) \geq a(b - c) \geq 0\), we get \(b^2(a - c)^2 \geq a^2(b - c)^2\), and hence \(b(a - c)^2 \geq a(b - c)^2\). Thus, it suffices to show that
\[b(3 + ab + ac - 7a^2) + a(3 + ab + bc - 7b^2) \geq 0.\]

This is true if
\[b(3 + ab - 7a^2) + a(3 + ab - 7b^2) \geq 0.\]

Indeed,
\[b(3 + ab - 7a^2) + a(3 + ab - 7b^2) = 3(a + b)(1 - 2ab) \geq 0,\]
since
\[1 - 2ab = (a - b)^2 + c^2 \geq 0.\]
The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

Second Solution. Without loss of generality, assume that \(a^2 + b^2 + c^2 = 1\) and \(a \leq b \leq c\).

Setting \(p = a + b + c\), \(q = ab + bc + ca\) and \(r = abc\), the inequality becomes
\[\sum \frac{5q - 11bc}{1 + bc} \leq 3,\]
\[3 \prod (1 + bc) + \sum (11bc - 5q)(1 + ca)(1 + ab) \geq 0,\]
\[3(1 + q + pr + r^2) + 11(q + 2pr + 3r^2) - 5q(3 + 2q + pr) \geq 0,\]
\[36r^2 + 5(5 - q)pr + 3 - q - 10q^2 \geq 0.\]

Since \(p^2 - 2q = 1\), the inequality has the homogeneous form
\[36r^2 + 5(5p^2 - 11q)pr + 3(p^2 - 2q)^3 - q(p^2 - 2q)^2 - 10q^2(p^2 - 2q) \geq 0.\]

According to P 2.57-(a) in Volume 1, for fixed \(p\) and \(q\), the product \(r = abc\) is minimal when \(b = c\) or \(a = 0\). Therefore, since \(5p^2 - 11q > 0\), it suffices to prove the inequality for \(a = 0\), and for \(b = c = 1\). For \(a = 0\), the original inequality becomes
\[-6bc \leq \frac{10bc}{b^2 + c^2 + bc} \leq 3,\]
which reduces to
\[(b - c)^2(3b^2 + 5bc + 3b^2) \geq 0,\]
while for \(b = c = 1\), we get
\[\frac{10a - 6}{a^2 + 3} + 2 \frac{5 - a}{a^2 + a + 2} \leq 3,\]
which is equivalent to
\[a(3a + 1)(a - 1)^2 \geq 0.\]

**Remark.** Similarly, we can prove the following generalization:
- Let \(a, b, c\) be nonnegative real numbers, no all are zero. If \(k > 0\), then
  \[
  \sum \frac{(2k + 3)a(b + c) + (k + 2)(k - 3)bc}{a^2 + b^2 + c^2 + kbc} \leq 3k,
  \]
with equality for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\(\square\)

**P 1.128.** Let \(a, b, c\) be nonnegative real numbers, no two of which are zero, and let
\[x = \frac{a^2 + b^2 + c^2}{ab + bc + ca}.\]

Prove that
\[
(a) \quad \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} + \frac{1}{2} \geq x + \frac{1}{x};
\]
\[
(b) \quad 6 \left( \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \right) \geq 5x + \frac{4}{x};
\]
\[
(c) \quad \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} - \frac{3}{2} \geq \frac{1}{3} \left( x - \frac{1}{x} \right).
\]

*(Vasile Cirtoaje, 2011)*

**Solution.** We will prove the more general inequality
\[
\frac{2a}{b + c} + \frac{2b}{c + a} + \frac{2c}{a + b} + 1 - 3k \geq (2 - k)x + \frac{2(1 - k)}{x},
\]
where \(0 \leq k \leq (21 + 6\sqrt{6})/25\). For \(k = 0\), \(k = 1/3\) and \(k = 4/3\), we get the inequalities in (a), (b) and (c), respectively. Let \(p = a + b + c\) and \(q = ab + bc + ca\). Since \(x = (p^2 - 2q)/q\), we can write the inequality as follows
\[
\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq f(p, q),
\]
\[
\sum_{cyc} \left( \frac{a}{b+c} + 1 \right) \geq 3 + f(p, q),
\]
\[
p\left(\frac{p^2 + q}{pq - abc}\right) \geq 3 + f(p, q).
\]

According to P 2.57-(a) in Volume 1, for fixed \(p\) and \(q\), the product \(abc\) is minimal when \(b = c\) or \(a = 0\). Therefore, it suffices to prove the inequality for \(a = 0\), and for \(b = c = 1\). For \(a = 0\), using the substitution \(y = b/c + c/b\), the desired inequality becomes

\[
2y + 1 - 3k \geq (2 - k)y + \frac{2(1 - k)}{y},
\]

\[
\frac{(y - 2)[k(y - 1) + 1]}{y} \geq 0.
\]

Since \(y \geq 2\), this inequality is clearly true. For \(b = c = 1\), the desired inequality becomes

\[
a + \frac{4}{a + 1} + 1 - 3k \geq \frac{(2 - k)(a^2 + 2)}{2a + 1} + \frac{2(1 - k)(2a + 1)}{a^2 + 2},
\]

which is equivalent to

\[
a(a - 1)^2[ka^2 + 3(1 - k)a + 6 - 4k] \geq 0.
\]

For \(0 \leq k \leq 1\), this is obvious, and for \(1 < k \leq (21 + 6\sqrt{5})/25\), we have

\[
ka^2 + 3(1 - k)a + 6 - 4k \geq [2\sqrt{k(6 - 4k) + 3(1 - k)}]a \geq 0.
\]

The equality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

**P 1.129.** If \(a, b, c\) are real numbers, then

\[
\frac{1}{a^2 + 7(b^2 + c^2)} + \frac{1}{b^2 + 7(c^2 + a^2)} + \frac{1}{c^2 + 7(a^2 + b^2)} \leq \frac{9}{5(a + b + c)^2}.
\]

_*(Vasile Cîrtoaje, 2008)_

**Solution.** Let \(p = a + b + c\) and \(q = ab + bc + ca\). Write the inequality as \(f_6(a, b, c) \geq 0\), where

\[
f_6(a, b, c) = 9 \prod \{a^2 + 7b^2 + 7c^2\} - 5p^2 \sum (b^2 + 7c^2 + 7a^2)(c^2 + 7a^2 + 7b^2).
\]

Since

\[
\prod \{a^2 + 7b^2 + 7c^2\} = \prod \{7(p^2 - 2q) - 6a^2\},
\]

\[
f_6(a, b, c) = 9 \prod \{7(p^2 - 2q) - 6a^2\} - 5p^2 \sum (b^2 + 7c^2 + 7a^2)(c^2 + 7a^2 + 7b^2).
\]

Since the inequality holds for \(a = b = c\), and for \(a = 0\) and \(b = c\) (or any cyclic permutation), the proof is complete.
\( f_6(a, b, c) \) has the highest coefficient
\[
A = 9(-6)^3 < 0.
\]
According to P 1.75 in Volume 1, it suffices to prove the original inequality for \( b = c = 1 \), when the inequality reduces to
\[
(a - 1)^2(a - 4)^2 \geq 0.
\]
Thus, the proof is completed. The equality holds for \( a = b = c \), and for \( a/4 = b = c \) (or any cyclic permutation).

\[\Box\]

**P 1.130.** If \( a, b, c \) are real numbers, then
\[
\frac{bc}{3a^2 + b^2 + c^2} + \frac{ca}{3b^2 + c^2 + a^2} + \frac{ab}{3c^2 + a^2 + b^2} \leq \frac{3}{5}.
\]
(Vasile Cîrtoaje and Pham Kim Hung, 2005)

**Solution.** Write the inequality as \( f_6(a, b, c) \geq 0 \), where
\[
f_6(a, b, c) = 3 \prod (3a^2 + b^2 + c^2) - 5 \sum bc(3b^2 + c^2 + a^2)(3c^2 + a^2 + b^2).
\]
Let \( p = a + b + c \) and \( q = ab + bc + ca \). From
\[
f_6(a, b, c) = 3 \prod (2a^2 + p^2 - 2q) - 5 \sum bc(2b^2 + p^2 - 2q)(2c^2 + p^2 - 2q),
\]
it follows that \( f_6(a, b, c) \) has the same highest coefficient \( A \) as
\[
24a^2b^2c^2 - 20 \sum b^3c^3;
\]
that is,
\[
A = 24 - 60 < 0.
\]
According to P 1.75 in Volume 1, it suffices to prove the original inequality for \( b = c = 1 \), when the inequality is equivalent to
\[
(a - 1)^2(3a - 2)^2 \geq 0.
\]
Thus, the proof is completed. The equality holds for \( a = b = c \), and for \( 3a/2 = b = c \) (or any cyclic permutation).

**Remark.** The inequality in P 1.130 is a particular case \( (k = 3) \) of the following more general result (Vasile Cîrtoaje, 2008):

- **Let** \( a, b, c \) **be real numbers. If** \( k > 1 \), **then**
\[
\sum \frac{k(k - 3)a^2 + 2(k - 1)bc}{ka^2 + b^2 + c^2} \leq \frac{3(k + 1)(k - 2)}{k + 2},
\]
**with** equality for \( a = b = c \), and for \( ka/2 = b = c \) (or any cyclic permutation).

\[\Box\]
P 1.131. If \(a, b, c\) are real numbers such that \(a + b + c = 3\), then

\[
\begin{align*}
(1) & \quad \frac{1}{2 + b^2 + c^2} + \frac{1}{2 + c^2 + a^2} + \frac{1}{2 + a^2 + b^2} \leq \frac{3}{4}; \\
(2) & \quad \frac{1}{8 + 5(b^2 + c^2)} + \frac{1}{8 + 5(c^2 + a^2)} + \frac{1}{8 + 5(a^2 + b^2)} \leq \frac{1}{6}.
\end{align*}
\]

(Vasile Cîrtoaje, 2006, 2009)

**Solution.** (a) Rewrite the inequality as follows

\[
\sum \left( \frac{1}{2 + b^2 + c^2} - \frac{1}{2} \right) \leq \frac{3}{4} - \frac{3}{2},
\]

\[
\sum \frac{b^2 + c^2}{2 + b^2 + c^2} \geq \frac{3}{2}.
\]

By the Cauchy-Schwarz inequality, we have

\[
\sum \frac{b^2 + c^2}{2 + b^2 + c^2} \geq \frac{(\sum \sqrt{b^2 + c^2})^2}{\sum (2 + b^2 + c^2)} = \sum a^2 + \sum \sqrt{(a^2 + b^2)(a^2 + c^2)}
\]

Thus, it suffices to show that

\[
2 \sum \sqrt{(a^2 + b^2)(a^2 + c^2)} \geq \sum a^2 + 9.
\]

Indeed, applying again the Cauchy-Schwarz inequality, we get

\[
2 \sum \sqrt{(a^2 + b^2)(a^2 + c^2)} \geq 2 \sum (a^2 + bc) = \sum a^2 + (\sum a)^2 = \sum a^2 + 9.
\]

The equality holds for \(a = b = c = 1\).

(b) Denote \(p = a + b + c\) and \(q = ab + bc + ca\), we write the inequality in the homogeneous form

\[
\frac{1}{8p^2 + 45(b^2 + c^2)} + \frac{1}{8p^2 + 45(c^2 + a^2)} + \frac{1}{8p^2 + 45(a^2 + b^2)} \leq \frac{1}{6p^2},
\]

which is equivalent to \(f_6(a, b, c) \geq 0\), where

\[
f_6(a, b, c) = \prod (53p^2 - 90q - 45a^2)
\]

\[
-6p^2 \sum (53p^2 - 90q - 45b^2)(53p^2 - 90q - 45c^2).
\]

Clearly, \(f_6(a, b, c)\) has the highest coefficient

\[
A = (-45)^3 < 0.
\]
By P 1.75 in Volume 1, it suffices to prove the original inequality for \( b = c \). In this case, the inequality is equivalent to
\[
(a - 1)^2(a - 13)^2 \geq 0.
\]
The equality holds for \( a = b = c = 1 \), and for \( a = 13/5 \) and \( b = c = 1/5 \) (or any cyclic permutation).

\[\square\]

**P 1.132.** If \( a, b, c \) are real numbers, then
\[
\frac{(a + b)(a + c)}{a^2 + 4(b^2 + c^2)} + \frac{(b + c)(b + a)}{b^2 + 4(c^2 + a^2)} + \frac{(c + a)(c + b)}{c^2 + 4(a^2 + b^2)} \leq \frac{4}{3}.
\]
(Vasile Cîrtoaje, 2008)

**Solution.** Let \( p = a + b + c \) and \( q = ab + bc + ca \). Write the inequality as \( f_6(a, b, c) \geq 0 \), where
\[
f_6(a, b, c) = 4 \prod (a^2 + 4b^2 + 4c^2) \]
\[
-3 \sum(a + b)(a + c)(b^2 + 4c^2 + 4a^2)(c^2 + 4a^2 + 4b^2)
\]
\[
= 4 \prod(4p^2 - 8q - 3a^2) - 3 \sum(a^2 + q)(4p^2 - 8q - 3b^2)(4p^2 - 8q - 3c^2).
\]
Thus, \( f_6(a, b, c) \) has the highest coefficient
\[
A = 4(-3)^3 - 3^4 < 0.
\]

By P 1.75 in Volume 1, it suffices to prove the original inequality for \( b = c = 1 \), when the inequality is equivalent to
\[
(a - 1)^2(2a - 7)^2 \geq 0.
\]
The equality holds for \( a = b = c \), and for \( 2a/7 = b = c \) (or any cyclic permutation).

\[\square\]

**P 1.133.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[
\sum \frac{1}{(b + c)(7a + b + c)} \leq \frac{1}{2(ab + bc + ca)}.
\]
(Vasile Cîrtoaje, 2009)
First Solution. Write the inequality as
\[
\sum \left[ 1 - \frac{4(ab + bc + ca)}{(b + c)(7a + b + c)} \right] \geq 1,
\]
\[
\sum \frac{(b - c)^2 + 3a(b + c)}{(b + c)(7a + b + c)} \geq 1.
\]

By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{(b - c)^2 + 3a(b + c)}{(b + c)(7a + b + c)} \geq \frac{(a + b + c)^4}{\sum [(b - c)^2 + 3a(b + c)](b + c)(7a + b + c)}.
\]

Therefore, it suffices to show that
\[
4(a + b + c)^4 \geq \sum (b^2 + c^2 - 2bc + 3ca + 3ab)(b + c)(7a + b + c).
\]

Write this inequality as
\[
\sum a^4 + abc \sum a + 3 \sum ab(a^2 + b^2) - 8 \sum a^2 b^2 \geq 0,
\]
\[
\sum a^4 + abc \sum a - \sum ab(a^2 + b^2) + 4 \sum ab(a - b)^2 \geq 0.
\]

Since \( \sum a^4 + abc \sum a - \sum ab(a^2 + b^2) \geq 0 \) (Schur’s inequality of degree four), the conclusion follows. The equality holds for \( a = b = c \), and also for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

Second Solution. Let \( p = a + b + c \) and \( q = ab + bc + ca \). We need to prove that
\[
f_6(a, b, c) = \prod (b + c)(7a + b + c)
\]
\[
-2(ab + bc + ca) \sum (a + b)(a + c)(7b + c + a)(7c + a + b)
\]
\[
= \prod (p - a)(p + 6a - 2q) \sum (p - b)(p - c)(p + 6b)(p + 6c).
\]

Clearly, \( f_6(a, b, c) \) has the highest coefficient \( A = (-6)^3 < 0 \). Thus, by P 2.76-(a) in Volume 1, it suffices to prove the original inequality for \( b = c = 1 \), and for \( a = 0 \). For \( b = c = 1 \), the inequality reduces to \( a(a - 1)^2 \geq 0 \), which is obviously true. For \( a = 0 \), the inequality can be written as
\[
\frac{1}{(b + c)^2} + \frac{1}{c(7b + c)} + \frac{1}{b(7c + b)} \leq \frac{1}{2bc},
\]
\[
\frac{1}{(b + c)^2} + \frac{b^2 + c^2 + 14bc}{bc[7(b^2 + c^2) + 50bc]} \leq \frac{1}{2bc},
\]
\[
\frac{1}{x + 2} + \frac{x + 14}{7x + 50} \leq \frac{1}{2},
\]
where \( x = b/c + c/b \), \( x \geq 2 \). This reduces to the obvious inequality \( (x - 2)(5x + 28) \geq 0 \).
P 1.134. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. Prove that

\[
\sum \frac{1}{b^2 + c^2 + 4a(b + c)} \leq \frac{9}{10(ab + bc + ca)}.
\]

(Vasile Cîrtoaje, 2009)

Solution. Let \(p = a + b + c\) and \(q = ab + bc + ca\). We need to prove that \(f_6(a, b, c) \geq 0\), where

\[
f_6(a, b, c) = 9 \prod [b^2 + c^2 + 4a(b + c)]
- 10(ab + bc + ca) \sum [a^2 + b^2 + 4c(a + b)][a^2 + c^2 + 4b(a + c)]
= 9 \prod (p^2 + 2q - a^2 - 4bc) - 10q \sum (p^2 + 2q - c^2 - 4ab)(p^2 + 2q - b^2 - 4ca).
\]

Clearly, \(f_6(a, b, c)\) has the same highest coefficient \(A\) as \(f(a, b, c)\), where

\[
f(a, b, c) = -9 \prod (a^2 + 4bc) = -9(65a^2b^2c^2 + 16abc \sum a^3 + 4 \sum a^3b^3);
\]

that is,

\[
A = -9(65 + 48 + 12) < 0.
\]

Thus, by P 2.76-(a) in Volume 1, it suffices to prove the original inequality for \(b = c = 1\), and for \(a = 0\). For \(b = c = 1\), the inequality reduces to \(a(a - 1)^2 \geq 0\), which is obviously true. For \(a = 0\), the inequality becomes

\[
\frac{1}{b^2 + c^2} + \frac{1}{b^2 + 4bc} + \frac{1}{c^2 + 4bc} \leq \frac{9}{10bc},
\]

\[
\frac{1}{b^2 + c^2} + \frac{b^2 + c^2 + 8bc}{4bc(b^2 + c^2) + 17b^2c^2} \leq \frac{9}{10bc},
\]

\[
\frac{1}{x} + \frac{x + 8}{4x + 17} \leq \frac{9}{10},
\]

where \(x = b/c + c/b\), \(x \geq 2\). The inequality is true, since it is equivalent to \((x - 2)(26x + 85) \geq 0\). The equality holds for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[
\square
\]

P 1.135. If \(a, b, c\) are nonnegative real numbers such that \(a + b + c = 3\), then

\[
\frac{1}{3 - ab} + \frac{1}{3 - bc} + \frac{1}{3 - ca} \leq \frac{9}{2(ab + bc + ca)}.
\]

(Vasile Cîrtoaje, 2011)
**First Solution.** We apply the SOS method. Write the inequality as

$$\sum \left( \frac{3}{2} - \frac{ab + bc + ca}{3 - bc} \right) \geq 0.$$  
$$\sum \frac{9 - 2(a + c) - 5bc}{3 - bc} \geq 0,$$
$$\sum \frac{a^2 + b^2 + c^2 - 3bc}{3 - bc} \geq 0.$$

Since

$$2(a^2 + b^2 + c^2 - 3bc) = 2(a^2 - bc) + 2(b^2 + c^2 - ab - ac) + 2(ab + ac - 2bc)$$
$$= (a - b)(a + c) + (a - c)(a + b) - 2b(a - b) - 2c(a - c) + 2c(a - b) + 2b(a - c)$$
$$= (a - b)(a - 2b + 3c) + (a - c)(a - 2c + 3b),$$

the required inequality is equivalent to

$$\sum \frac{(a - b)(a - 2b + 3c) + (a - c)(a - 2c + 3b)}{3 - bc} \geq 0,$$
$$\sum \frac{(a - b)(a - 2b + 3c)}{3 - bc} + \sum \frac{(b - a)(b - 2a + 3c)}{3 - ca} \geq 0,$$
$$\sum \frac{(a - b)^2[9 - c(a + b + 3c)]}{(3 - bc)(3 - ca)} \geq 0,$$
$$\sum (a - b)^2(3 - ab)(3 + c)(3 - 2c) \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c$. Then it suffices to prove that

$$(b - c)^2(3 - bc)(3 + a)(3 - 2a) + (c - a)^2(3 - ca)(3 + b)(3 - 2b) \geq 0,$$

which is equivalent to

$$(a - c)^2(3 - ac)(3 + b)(3 - 2b) \geq (b - c)^2(3 - bc)(a + 3)(2a - 3).$$

Since $3 - 2b = a - b + c \geq 0$, we can obtain this inequality by multiplying the inequalities

$$b^2(a - c)^2 \geq a^2(b - c)^2,$$
$$a(3 - ac) \geq b(3 - bc) \geq 0,$$
$$a(3 + b)(3 - 2b) \geq b(a + 3)(2a - 3) \geq 0.$$

We have

$$a(3 - ac) - b(3 - bc) = (a - b)[3 - c(a + b)] = (a - b)(3 - 3c + c^2)$$
$$\geq 3(a - b)(1 - c) \geq 0.$$
Also, since \( a + b \leq a + b + c = 3 \), we have

\[
a(3 + b)(3 - 2b) - b(a + 3)(2a - 3) = 9(a + b) - 6ab - 2ab(a + b) \geq 9(a + b) - 12ab \geq 3(a + b)^2 - 12ab = 3(a - b)^2 \geq 0.
\]

The equality holds for \( a = b = c = 1 \), and for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

**Second Solution.** Let \( p = a + b + c \) and \( q = ab + bc + ca \). We need to prove that \( f_6(a, b, c) \geq 0 \), where

\[
f_6(a, b, c) = 3 \prod (p^2 - 3bc) - 2q \sum (p^2 - 2ca)(p^2 - 2ab).
\]

Clearly, \( f_6(a, b, c) \) has the highest coefficient

\[A = 3(-3)^3 < 0.\]

Thus, by P 2.76-(a) in Volume 1, it suffices to prove the original inequality for \( b = c \), and for \( a = 0 \). For \( b = c = 3 - a \), the inequality reduces to

\[a(9 - a)(a - 1)^2 \geq 0,
\]

which is obviously true. For \( a = 0 \), which yields \( b + c = 3 \), the inequality can be written as

\[(9 - 4bc)(9 - bc) \geq 0.
\]

Indeed,

\[(9 - 4bc)(9 - bc) = (b - c)^2(b^2 + c^2 + bc) \geq 0.
\] \(\square\)

**P 1.136.** If \( a, b, c \) are nonnegative real numbers such that \( a + b + c = 3 \), then

\[
\frac{bc}{a^2 + a + 6} + \frac{ca}{b^2 + b + 6} + \frac{ab}{c^2 + c + 6} \leq \frac{3}{8}.
\]

*(Vasile Cîrtoaje, 2009)*

**Solution.** Write the inequality as

\[
\sum \frac{bc}{3a^2 + ap + 2p^2} \leq \frac{1}{8},
\]

where \( p = a + b + c \). We need to prove that \( f_6(a, b, c) \geq 0 \), where

\[
f_6(a, b, c) = \prod (3a^2 + ap + 2p^2) - 8 \sum bc(3b^2 + bp + 2p^2)(3c^2 + cp + 2p^2).
\]
Clearly, $f_6(a, b, c)$ has the same highest coefficient as

$$27a^2b^2c^2 - 72\sum b^3c^3,$$

that is,

$$A = 27 - 216 < 0.$$

Thus, by P 2.76-(a) in Volume 1, it suffices to prove that $f_6(a, 1, 1) \geq 0$ and $f_6(0, b, c) \geq 0$ for all $a, b, c \geq 0$. Indeed, we have

$$f_6(a, 1, 1) = 2a(a^2 + 9a + 3)(a - 1)^2(6a + 1) \geq 0$$

and

$$f_6(0, b, c) = 2(b - c)^2(5b^2 + 5bc + 2c^2)(2b^2 + 5bc + 5c^2) \geq 0.$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

\[\square\]

**P 1.137.** If $a, b, c$ are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{1}{8a^2 - 2bc + 21} + \frac{1}{8b^2 - 2ca + 21} + \frac{1}{8c^2 - 2ab + 21} \geq \frac{1}{9}.$$

*(Michael Rozenberg, 2013)*

**Solution.** Let

$$q = ab + bc + ca.$$

Write the inequality in the homogeneous form $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = 3q \sum (8b^2 - 2ca + 7q)(8c^2 - 2ab + 7q) - \prod (8a^2 - 2bc + 7q).$$

Clearly, $f_6(a, b, c)$ has the same highest coefficient as $f(a, b, c)$, where

$$f(a, b, c) = -8 \prod (4a^2 - bc) = -8(63a^2b^2c^2 - 16 \sum a^3b^3 + 4abc \sum a^2);$$

that is,

$$A = -8(63 - 48 + 12) < 0.$$

By P 2.76-(a) in Volume 1, it suffices to prove that $f_6(a, 1, 1) \geq 0$ and $f_6(0, b, c) \geq 0$ for all $a, b, c \geq 0$. Indeed, we have

$$f_6(a, 1, 1) = 0$$

and

$$f_6(0, b, c) = 8b^2c^2(b - c)^2 \geq 0.$$
The equality holds when two of \( a, b, c \) are equal.

**Remark.** The following identity holds:

\[
\sum \frac{9}{8a^2 - 2bc + 21} - 1 = \frac{8 \prod (a - b)^2}{\prod (a^2 - 2bc + 21)}.
\]

\[\square\]

**P 1.138.** Let \( a, b, c \) be real numbers, no two of which are zero. Prove that

\( (a) \)

\[
\frac{a^2 + bc}{b^2 + c^2} + \frac{b^2 + ca}{c^2 + a^2} + \frac{c^2 + ab}{a^2 + b^2} \geq \frac{(a + b + c)^2}{a^2 + b^2 + c^2};
\]

\( (b) \)

\[
\frac{a^2 + 3bc}{b^2 + c^2} + \frac{b^2 + 3ca}{c^2 + a^2} + \frac{c^2 + 3ab}{a^2 + b^2} \geq \frac{6(ab + bc + ca)}{a^2 + b^2 + c^2}.
\]

(\textit{Vasile Cirtoaje, 2014})

**Solution.** (a) Using the known inequality

\[
\sum \frac{a^2}{b^2 + c^2} \geq \frac{3}{2}
\]

and the Cauchy-Schwarz inequality yields

\[
\sum \frac{a^2 + bc}{b^2 + c^2} = \sum \frac{a^2}{b^2 + c^2} + \sum \frac{bc}{b^2 + c^2} \geq \sum \frac{1}{2} \left( \frac{bc}{b^2 + c^2} \right) = \sum \frac{(b + c)^2}{2(b^2 + c^2)} \geq \frac{1}{2} \sum \frac{(b + c)^2}{2(b^2 + c^2)} = \frac{(a + b + c)^2}{a^2 + b^2 + c^2}.
\]

The equality holds for \( a = b = c \).

(b) We have

\[
\sum \frac{a^2 + 3bc}{b^2 + c^2} = \sum \frac{a^2}{b^2 + c^2} + \sum \frac{3bc}{b^2 + c^2} \geq \frac{3}{2} + \sum \frac{3bc}{b^2 + c^2}
\]

\[
= -3 + 3 \sum \left( \frac{1}{2} + \frac{bc}{b^2 + c^2} \right) = -3 + 3 \sum \frac{(b + c)^2}{2(b^2 + c^2)}
\]

\[
\geq -3 + \frac{3 \left[ \sum (b + c)^2 \right]}{2 \sum (b^2 + c^2)} = -3 + \frac{3 \left( \sum a^2 \right)^2}{\sum a^2} = \frac{6(ab + bc + ca)}{a^2 + b^2 + c^2}.
\]

The equality holds for \( a = b = c \).
P 1.139. Let $a, b, c$ be real numbers such that $ab + bc + ca \geq 0$ and no two of which are zero. Prove that

$$\frac{a(b + c)}{b^2 + c^2} + \frac{b(c + a)}{c^2 + a^2} + \frac{c(a + b)}{a^2 + b^2} \geq \frac{3}{10}$$

(Vasile Cîrtoaje, 2014)

**Solution.** Since the problem remains unchanged by replacing $a, b, c$ by $-a, -b, -c$, it suffices to consider the cases $a, b, c \geq 0$ and $a < 0, b \geq 0, c \geq 0$.

**Case 1:** $a, b, c \geq 0$. We have

$$\sum \frac{a(b + c)}{b^2 + c^2} \geq \sum \frac{a(b + c)}{(b + c)^2} = \sum \frac{a}{b + c} \geq \frac{3}{2} > \frac{3}{10}.$$

**Case 2:** $a < 0, b \geq 0, c \geq 0$. Replacing $a$ by $-a$, we need to show that

$$\frac{b(c - a)}{a^2 + c^2} + \frac{c(b - a)}{a^2 + b^2} - \frac{a(b + c)}{b^2 + c^2} \geq \frac{3}{10}$$

for any nonnegative numbers $a, b, c$ such that

$$a \leq \frac{bc}{b + c}.$$

We show first that

$$\frac{b(c - a)}{a^2 + c^2} \geq \frac{b(c - x)}{x^2 + c^2},$$

where $x = \frac{bc}{b + c}, x \geq a$. This is equivalent to

$$b(x - a)((c - a)x + ac + c^2) \geq 0,$$

which is true because

$$(c - a)x + ac + c^2 = \frac{c^2(a + 2b + c)}{b + c} \geq 0.$$

Similarly, we can show that

$$\frac{c(b - a)}{a^2 + b^2} \geq \frac{c(b - x)}{x^2 + b^2}.$$

In addition,

$$\frac{a(b + c)}{b^2 + c^2} \leq \frac{x(b + c)}{b^2 + c^2}.$$
Therefore, it suffices to prove that
\[
\frac{b(c-x)}{x^2+c^2} + \frac{c(b-x)}{x^2+b^2} = \frac{x(b+c)}{b^2+c^2} \geq \frac{3}{10}.
\]
Denote
\[
p = \frac{b}{b+c}, \quad q = \frac{c}{b+c}, \quad p + q = 1.
\]
Since
\[
\frac{b(c-x)}{x^2+c^2} = \frac{p}{1+p^2}, \quad \frac{c(b-x)}{x^2+b^2} = \frac{q}{1+q^2}
\]
and
\[
\frac{x(b+c)}{b^2+c^2} = \frac{bc}{b^2+c^2} = \frac{pq}{1-2pq},
\]
we need to show that
\[
\frac{p}{1+p^2} + \frac{q}{1+q^2} - \frac{pq}{1-2pq} \geq \frac{3}{10}.
\]
Since
\[
\frac{p}{1+p^2} + \frac{q}{1+q^2} = \frac{1+pq}{2-2pq+p^2q^2},
\]
the inequality can be written as
\[
(pq+2)(1-4pq) \geq 0,
\]
which is true since
\[
1-4pq = (p+q)^2 - 4pq = (p-q)^2 \geq 0.
\]
The equality holds for $-2a = b = c$ (or any cyclic permutation).

\[\square\]

**P 1.140.** If $a, b, c$ are positive real numbers such that $abc > 1$, then
\[
\frac{1}{a+b+c-3} + \frac{1}{abc-1} \geq \frac{4}{ab+bc+ca-3}.
\]
(\textit{Vasile Cîrtoaje, 2011})

**Solution (by Vo Quoc Ba Can).** By the AM-GM inequality, we have
\[
a + b + c \geq 3\sqrt[3]{abc} > 3,
\]
\[
ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} > 3.
\]
Without loss of generality, assume that $a = \min\{a, b, c\}$. By the Cauchy-Schwarz inequality, we have

$$\left(\frac{1}{a + b + c - 3} + \frac{1}{abc - 1}\right)\left[a(a + b + c - 3) + \frac{abc - 1}{a}\right] \geq \left(\sqrt{a} + \frac{1}{\sqrt{a}}\right)^2.$$  

Therefore, it suffices to prove that

$$\frac{(a + 1)^2}{4a} \geq \frac{a(a + b + c - 3) + \frac{abc - 1}{a}}{ab + bc + ca - 3}.$$  

Since

$$a(a + b + c - 3) + \frac{abc - 1}{a} = ab + bc + ca - 3 + \frac{(a - 1)^3}{a},$$

this inequality can be written as follows

$$\frac{(a + 1)^2}{4a} - 1 \geq \frac{(a - 1)^3}{a(ab + bc + ca - 3)},$$

$$\frac{(a - 1)^2}{4a} \geq \frac{(a - 1)^3}{a(ab + bc + ca - 3)},$$

$$(a - 1)^2(ab + bc + ca + 1 - 4a) \geq 0.$$  

This is true since

$$bc \geq \sqrt[3]{(abc)^2} > 1,$$

and hence

$$ab + bc + ca + 1 - 4a > a^2 + 1 + a^2 + 1 - 4a = 2(a - 1)^2 \geq 0.$$  

The equality holds for $a > 1$ and $b = c = 1$ (or any cyclic permutation).

**Remark.** Using this inequality, we can prove P 2.84 in Volume 1, which states that

$$(a + b + c - 3)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 3\right) + abc + \frac{1}{abc} \geq 2$$

for any positive real numbers $a, b, c$. This inequality is clearly true for $abc = 1$. In addition, it remains unchanged by substituting $a, b, c$ with $1/a, 1/b, 1/c$, respectively. Therefore, it suffices to consider the case $abc > 1$. Since $a + b + c \geq 3\sqrt[3]{abc} > 3$, we can write the required inequality as $E \geq 0$, where

$$E = ab + bc + ca - 3abc + \frac{(abc - 1)^2}{a + b + c - 3}.$$
According to the inequality in P 1.140, we have
\[
E \geq ab + bc + ca - 3abc + (abc - 1)^2 \left( \frac{4}{ab + bc + ca - 3} - \frac{1}{abc - 1} \right)
\]
\[
= (ab + bc + ca - 3) + \frac{4(abc - 1)^2}{ab + bc + ca - 3} - 4(abc - 1)
\]
\[
\geq 2\sqrt{(ab + bc + ca - 3) \cdot \frac{4(abc - 1)^2}{ab + bc + ca - 3} - 4(abc - 1)} = 0.
\]
\[
\square
\]

**P 1.141.** Let \(a, b, c\) be positive real numbers, no two of which are zero. Prove that
\[
\sum \frac{(4b^2 - ac)(4c^2 - ab)}{b + c} \leq \frac{27}{2} abc.
\]

*(Vasile Cîrtoaje, 2011)*

**Solution.** Since
\[
\sum \frac{(4b^2 - ac)(4c^2 - ab)}{b + c} = \sum \frac{bc(16bc + a^2)}{b + c} - 4 \sum \frac{a(b^3 + c^3)}{b + c}
\]
\[
= \sum \frac{bc(16bc + a^2)}{b + c} - 4 \sum \frac{a(b^2 + c^2)}{b + c} + 12abc
\]
\[
= \sum bc \left[ \frac{a^2}{b + c} + \frac{16bc}{b + c} - 4(b + c) \right] + 12abc
\]
\[
= \sum bc \left[ \frac{a^2}{b + c} - 4\frac{(b - c)^2}{b + c} \right] + 12abc
\]
we can write the inequality as follows
\[
\sum bc \left[ \frac{a}{2} - \frac{a^2}{b + c} + \frac{4(b - c)^2}{b + c} \right] \geq 0,
\]
\[
8 \sum \frac{bc(b - c)^2}{b + c} \geq abc \sum \frac{2a - b - c}{b + c}.
\]

In addition, since
\[
\sum \frac{2a - b - c}{b + c} = \sum \frac{(a - b) + (a - c)}{b + c} = \sum \frac{a - b}{b + c} + \sum \frac{b - a}{c + a}
\]
\[
= \sum \frac{(a - b)^2}{(b + c)(c + a)} = \sum \frac{(b - c)^2}{(c + a)(a + b)},
\]
the inequality can be restated as
\[
8 \sum \frac{bc(b-c)^2}{b+c} \geq abc \sum \frac{(b-c)^2}{(c+a)(a+b)},
\]
\[
\sum \frac{bc(b-c)^2(8a^2 + 8bc + 7ab + 7ac)}{(a+b)(b+c)(c+a)} \geq 0.
\]
Since the last form is obvious, the proof is completed. The equality holds for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\[\square\]

P 1.142. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero, such that
\[a + b + c = 3.\]
Prove that
\[\frac{a}{3a + bc} + \frac{b}{3b + ca} + \frac{c}{3c + ab} \geq \frac{2}{3}.\]

Solution. Since
\[3a + bc = a(a + b + c) + bc = (a + b)(a + c),\]
we can write the inequality as follows
\[a(b + c) + b(c + a) + c(a + b) \geq \frac{2}{3}(a + b)(b + c)(c + a),\]
\[6(ab + bc + ca) \geq 2[(a + b + c)(ab + bc + ca) - abc],\]
\[2abc \geq 0.\]
The equality holds for \(a = 0\), or \(b = 0\), or \(c = 0\).

\[\square\]

P 1.143. Let \(a, b, c\) be positive real numbers such that
\[(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 10.\]
Prove that
\[\frac{19}{12} \leq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leq \frac{5}{3}.\]

\(\text{(Vasile Cîrtoaje, 2012)}\)
**First Solution.** Write the hypothesis

\[(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 10\]

as

\[\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} = 7\]

and

\[(a + b)(b + c)(c + a) = 9abc.\]

Using the substitutions \(x = \frac{b + c}{a}, \ y = \frac{c + a}{b}\) and \(z = \frac{a + b}{c}\), we need to show that \(x + y + z = 7\) and \(xyz = 9\) involve

\[\frac{19}{12} \leq \frac{1}{x} \leq \frac{5}{3}\]

or, equivalently,

\[\frac{19}{12} \leq \frac{1}{x} + \frac{x(7-x)}{9} \leq \frac{5}{3}\]

Clearly, \(x, y, z \in (0, 7)\). The left inequality is equivalent to

\[(x - 4)(2x - 3)^2 \leq 0,\]

while the right inequality is equivalent to

\[(x - 1)(x - 3)^2 \geq 0.\]

These inequalities are true if \(1 \leq x \leq 4\). To show that \(1 \leq x \leq 4\), from \((y + z)^2 \geq 4yz\), we get

\[(7 - x)^2 \geq \frac{36}{x},\]

\[(x - 1)(x - 4)(x - 9) \geq 0,\]

\[1 \leq x \leq 4.\]

Thus, the proof is completed. The left inequality is an equality for \(2a = b = c\) (or any cyclic permutation), and the right inequality is an equality for \(a/2 = b = c\) (or any cyclic permutation).

**Second Solution.** Due to homogeneity, assume that \(b + c = 2\); this involves \(bc \leq 1\). From the hypothesis

\[(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 10,\]

we get

\[bc = \frac{2a(a + 2)}{9a - 2}.\]
Since 
\[ \frac{bc - 1}{9a - 2} = \frac{(a - 2)(2a - 1)}{9a - 2}, \]
from the condition \( bc \leq 1 \), we get
\[ \frac{1}{2} \leq a \leq 2. \]

We have
\[ \frac{b}{c + a} + \frac{c}{a + b} = \frac{a(b + c) + b^2 + c^2}{a^2 + (b + c)a + bc} = \frac{2a + 4 - 2bc}{a^2 + 2a + bc} = \frac{2(7a^2 + 12a - 4)}{9a^2(a + 2)} = \frac{2(7a - 2)}{9a^2}, \]
and hence
\[ \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} = \frac{a}{2} + \frac{2(7a - 2)}{9a^2} = \frac{9a^3 + 28a - 8}{18a^2}. \]
Thus, we need to show that
\[ \frac{19}{12} \leq \frac{9a^3 + 28a - 8}{18a^2} \leq \frac{5}{3}. \]
These inequalities are true, since the left inequality is equivalent to
\[ (2a - 1)(3a - 4)^2 \geq 0, \]
and the right inequality is equivalent to
\[ (a - 2)(3a - 2)^2 \leq 0. \]

Remark. Similarly, we can prove the following generalization.

- Let \( a, b, c \) be positive real numbers such that
  \[ (a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 9 + \frac{8k^2}{1 - k^2}, \]
where \( k \in (0, 1) \). Then,
  \[ \frac{k^2}{1 + k} \leq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leq \frac{3}{2} \leq \frac{k^2}{1 - k}. \]
\[ \Box \]
P 1.144. Let $a, b, c$ be nonnegative real numbers, no two of which are zero, such that $a + b + c = 3$. Prove that

$$\frac{9}{10} < \frac{a}{2a + bc} + \frac{b}{2b + ca} + \frac{c}{2c + ab} \leq 1.$$  

(Vasile Cîrtoaje, 2012)

Solution. (a) Since

$$\frac{a}{2a + bc} - \frac{1}{2} = \frac{-bc}{2(2a + bc)},$$

we can write the right inequality as

$$\sum \frac{bc}{2a + bc} \geq 1.$$  

According to the Cauchy-Schwarz inequality, we have

$$\sum \frac{bc}{2a + bc} \geq \frac{(\sum bc)^2}{\sum bc(2a + bc)} = \frac{\sum b^2c^2 + 2abc \sum a}{6abc + \sum b^2c^2} = 1.$$  

The equality holds for $a = b = c = 1$, and also for $a = 0$, or $b = 0$, or $c = 0$.

(b) First Solution. For the nontrivial case $a, b, c > 0$, we can write the left inequality as

$$\sum \frac{1}{2 + \frac{bc}{a}} > \frac{9}{10}.$$  

Using the substitutions

$$x = \sqrt{\frac{bc}{a}}, \quad y = \sqrt{\frac{ca}{b}}, \quad z = \sqrt{\frac{ab}{c}},$$

we need to show that

$$\sum \frac{1}{2 + x^2} > \frac{9}{10}$$

for all positive real numbers $x, y, z$ satisfying $xy + yz + zx = 3$. By expanding, the inequality becomes

$$4 \sum x^2 + 48 \geq 9x^2y^2z^2 + 8 \sum x^2y^2.$$  

Since

$$\sum x^2y^2 = (\sum xy)^2 - 2xyz \sum x = 9 - 2xyz \sum x,$$

we can write the desired inequality as

$$4 \sum x^2 + 16xyz \sum x \geq 9x^2y^2z^2 + 24,$$
which is equivalent to
\[ 16xyz(x + y + z) \geq 9x^2y^2z^2 + 4(2xy + 2yz + 2zx - x^2 - y^2 - z^2). \]

Using Schur’s inequality
\[ \frac{9xyz}{x + y + z} \geq 2xy + 2yz + 2zx - x^2 - y^2 - z^2, \]
it suffices to prove that
\[ 16xyz(x + y + z) \geq 9x^2y^2z^2 + \frac{36xyz}{x + y + z}. \]

This is true if
\[ 16(x + y + z) \geq 9xyz + \frac{36}{x + y + z}. \]

Since
\[ x + y + z \geq \sqrt{3(xy + yz + zx)} = 3 \]
and
\[ 1 = \frac{xy + yz + zx}{3} \geq \sqrt[3]{xyz}, \]
we have
\[ 16(x + y + z) - 9xyz - \frac{36}{x + y + z} \geq 48 - 9xyz - 12 = 9(4 - xyz) > 0. \]

**Second Solution.** As it is shown at the first solution, it suffices to show that
\[ \sum \frac{1}{2 + x^2} > \frac{9}{10} \]
for all positive real numbers \( x, y, z \) satisfying \( xy + yz + zx = 3 \). Rewrite this inequality as
\[ \sum \frac{x^2}{2 + x^2} < \frac{6}{5}. \]

Let \( p \) and \( q \) be two positive real numbers such that \( p + q = \sqrt{3} \).

By the Cauchy-Schwarz inequality, we have
\[ \frac{x^2}{2 + x^2} = \frac{3x^2}{2(xy + yz + zx) + 3x^2} \leq \frac{(px + qx)^2}{2p^2x + p^2x + q^2x^2 + q^2x^2 + x^2 + 2yz}. \]
Therefore,
\[ \sum \frac{x^2}{2 + x^2} \leq \sum \frac{p^2 x}{2(x + y + z)} + \sum \frac{q^2 x^2}{x^2 + 2yz} = \frac{p^2}{2} + q^2 \sum \frac{x^2}{x^2 + 2yz}. \]

Thus, it suffices to prove that
\[ \frac{p^2}{2} + q^2 \sum \frac{x^2}{x^2 + 2yz} < \frac{6}{5}. \]

We claim that
\[ \sum \frac{x^2}{x^2 + 2yz} < 2. \]

Under this assumption, we only need to show that
\[ \frac{p^2}{2} + 2q^2 \leq \frac{6}{5}. \]

Indeed, choosing \( p = \frac{4\sqrt{3}}{5} \) and \( q = \frac{\sqrt{3}}{5} \), we have \( p + q = \sqrt{3} \) and \( \frac{p^2}{2} + 2q^2 = \frac{6}{5} \). To complete the proof, we need to prove the homogeneous inequality \( \sum \frac{x^2}{x^2 + 2yz} < 2 \), which is equivalent to
\[ \sum \frac{yz}{x^2 + 2yz} > \frac{1}{2}. \]

By the Cauchy-Schwarz inequality, we get
\[ \sum \frac{yz}{x^2 + 2yz} \geq \left( \sum \frac{yz}{x^2 + 2yz} \right) \left( \sum \frac{yz}{x^2 + 2yz} \right) = \sum \frac{y^2 z^2}{xyz} + 2xyz \sum \frac{x}{x^2 + 2yz} > \frac{1}{2}. \]

\[ \square \]

**P 1.145.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. Prove that
\[ \frac{a^3}{2a^2 + bc} + \frac{b^3}{2b^2 + ca} + \frac{c^3}{2c^2 + ab} \leq \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}. \]

*(Vasile Cîrtoaje, 2011)*

**Solution.** Write the inequality as follows
\[ \sum \left[ \frac{a^3}{a^2 + b^2 + c^2} - \frac{a^3}{2a^2 + bc} \right] \geq 0, \]
\[ \sum \frac{a^3(a^2 + bc - b^2 - c^2)}{2a^2 + bc} \geq 0, \]
\[ \sum \frac{a^3[a^2(b + c) - b^3 - c^3]}{(b + c)(2a^2 + bc)} \geq 0, \]
\[ \sum \frac{a^3b(a^2 - b^2) + a^3c(a^2 - c^2)}{(b + c)(2a^2 + bc)} \geq 0, \]
\[ \sum \frac{a^3b(a^2 - b^2)}{(b + c)(2a^2 + bc)} + \sum \frac{a^3c(a^2 - c^2)}{(b + c)(2a^2 + bc)} \geq 0, \]
\[ \sum \frac{a^3b(a^2 - b^2)}{(b + c)(2a^2 + bc)} + \sum \frac{b^3(a^2 - a^2)}{(c + a)(2b^2 + ca)} \geq 0, \]
\[ \sum \frac{ab(a + b)(a - b)^2[2a^2b^2 + c(a^3 + a^2b + ab^2 + b^3) + c^2(a^2 + ab + b^2)]}{(b + c)(c + a)(2a^2 + bc)(2b^2 + ca)} \geq 0. \]

The equality holds for \( a = b = c \), and also for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

\[ \square \]

**P 1.146.** Let \( a, b, c \) be positive real numbers, no two of which are zero. Prove that

\[ \frac{a^3}{4a^2 + bc} + \frac{b^3}{4b^2 + ca} + \frac{c^3}{4c^2 + ab} \geq \frac{a + b + c}{5}. \]

*(Vasile Cîrtoaje, 2011)*

**Solution.** Assume that \( a \geq b \geq c \), and write the inequality as follows

\[ \sum \left( \frac{a^3}{4a^2 + bc} - \frac{a}{5} \right) \geq 0, \]
\[ \sum \frac{a(a^2 - bc)}{4a^2 + bc} \geq 0, \]
\[ \sum \frac{a[(a - b)(a + c) + (a - c)(a + b)]}{4a^2 + bc} \geq 0, \]
\[ \sum \frac{a(a - b)(a + c)}{4a^2 + bc} + \sum \frac{a(a - c)(a + b)}{4a^2 + bc} \geq 0, \]
\[ \sum \frac{a(a - b)(a + c)}{4a^2 + bc} + \sum \frac{b(b - a)(b + c)}{4b^2 + ca} \geq 0, \]
\[ \sum \frac{c(a - b)^2[(a - b)^2 + bc + ca - ab]}{(4a^2 + bc)(4b^2 + ca)} \geq 0. \]
Clearly, it suffices to show that
\[
\sum \frac{c(a-b)^2(bc+ca-ab)}{(4a^2+bc)(4b^2+ca)} \geq 0,
\]
we can be written as
\[
\sum (a-b)^2(bc+ca-ab)(4c^3+abc) \geq 0.
\]
Since \(ca+ab-bc>0\), it is enough to prove that
\[
(c-a)^2(ab+bc-ca)(4b^3+abc) + (a-b)^2(bc+ca-ab)(4c^3+abc) \geq 0.
\]
In addition, since \((c-a)^2 \geq (a-b)^2\), \(4b^3+abc \geq 4c^3+abc\) and \(ab+bc-ca>0\), we only need to show that
\[
(a-b)^2(ab+bc-ca)(4c^3+abc) + (a-b)^2(bc+ca-ab)(4c^3+abc) \geq 0.
\]
This is equivalent to the obvious inequality
\[
abc(a-b)^2(4c^3+bc) \geq 0.
\]
The equality holds for \(a=b=c\).

\[\square\]

**P 1.147.** If \(a, b, c\) are positive real numbers, then
\[
\frac{1}{(2+a)^2} + \frac{1}{(2+b)^2} + \frac{1}{(2+c)^2} \geq \frac{3}{6+ab+bc+ca}.
\]
*(Vasile Cîrtoaje, 2013)*

**Solution.** By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{1}{(2+a)^2} \geq \frac{4(a+b+c)^2}{\sum (2+a)^2(2+b)^2}.
\]
Thus, it suffices to show that
\[
4(a+b+c)^2(6+ab+bc+ca) \geq \sum (2+a)^2(2+b)^2.
\]
This inequality is equivalent to
\[
2p^2q - 3q^2 + 3pr + 12q \geq 6(pq + 3r),
\]
where
\[ p = a + b + c, \quad q = ab + bc + ca, \quad r = abc. \]

According to AM-GM inequality,
\[ 2p^2q - 3q^2 + 3pr + 12q \geq 2\sqrt{12q(2p^2q - 3q^2 + 3pr)}. \]

Therefore, it is enough to prove the homogeneous inequality
\[ 4q(2p^2q - 3q^2 + 3pr) \geq 3(pq + 3r)^2, \]
which can be written as
\[ 5p^2q^2 \geq 12q^3 + 6pqr + 27r^2. \]

Since \( pq \geq 9r \), we have
\[ 3(5p^2q^2 - 12q^3 - 6pqr - 27r^2) \geq 15p^2q^2 - 36q^3 - 2p^2q^2 - p^2q^2 = 12q^2(p^2 - 3q) \geq 0. \]

The equality holds for \( a = b = c = 1 \). \( \square \)

**P 1.148.** If \( a, b, c \) are positive real numbers, then
\[
\frac{1}{1+3a} + \frac{1}{1+3b} + \frac{1}{1+3c} \geq \frac{3}{3 + abc}. 
\]

*(Vasile Cîrtoaje, 2013)*

**Solution.** Set
\[ p = a + b + c, \quad q = ab + bc + ca, \quad r = \sqrt[3]{abc}, \]
and write the inequality as follows
\[
(3 + r^3) \sum (1 + 3b)(1 + 3c) \geq 3(1 + 3a)(1 + 3b)(1 + 3c),
\]
\[
(3 + r^3)(3 + 6p + 9q) \geq 3(1 + 3p + 9q + 27r^3),
\]
\[
r^3(2p + 3q) + 2 + 3p \geq 26r^3.
\]

By virtue of the AM-GM inequality, we have
\[ p \geq 3r, \quad q \geq 3r^2. \]

Therefore, it suffices to show that
\[ r^3(6r + 9r^2) + 2 + 9r \geq 26r^3, \]
which is equivalent to the obvious inequality

$$(r-1)^2(9r^3+24r^2+13r+2) \geq 0.$$ 

The equality holds for $a = b = c = 1$. 

\(\square\)

**P 1.149.** Let $a, b, c$ be real numbers, no two of which are zero. If $1 \leq k \leq 3$, then

$$
\left( k + \frac{2ab}{a^2 + b^2} \right) \left( k + \frac{2bc}{b^2 + c^2} \right) \left( k + \frac{2ca}{c^2 + a^2} \right) \geq (k-1)(k^2-1).
$$

*(Vasile Cîrtoaje and Vo Quoc Ba Can, 2011)*

**Solution.** If $a, b, c$ are the same sign, then the inequality is obvious since

$$
\left( k - \frac{2ab}{a^2 + b^2} \right) \left( k - \frac{2bc}{b^2 + c^2} \right) \left( k - \frac{2ca}{c^2 + a^2} \right) \geq (k-1)(k^2-1).
$$

Since the inequality remains unchanged by replacing $a, b, c$ with $-a, -b, -c$, it suffices to consider further that $a \leq 0$, $b \geq 0$, $c \geq 0$. Setting $-a$ for $a$, we need to show that

$$
\left( k - \frac{2ab}{a^2 + b^2} \right) \left( k + \frac{2bc}{b^2 + c^2} \right) \left( k - \frac{2ca}{c^2 + a^2} \right) \geq (k-1)(k^2-1)
$$

for $a, b, c \geq 0$. Since

$$
\left( k - \frac{2ab}{a^2 + b^2} \right) \left( k - \frac{2ca}{c^2 + a^2} \right) = \left[ k - 1 + \frac{(a-b)^2}{a^2 + b^2} \right] \left[ k - 1 + \frac{(a-c)^2}{c^2 + a^2} \right]
$$

$$
\geq (k-1)^2 + (k-1) \left[ \frac{(a-b)^2}{a^2 + b^2} + \frac{(a-c)^2}{c^2 + a^2} \right],
$$

it suffices to prove that

$$
\left[ k - 1 + \frac{(a-b)^2}{a^2 + b^2} + \frac{(a-c)^2}{c^2 + a^2} \right] \left( k + \frac{2bc}{b^2 + c^2} \right) \geq k^2 - 1.
$$

According to the inequality (a) from P 1.19 in Volume 5, we have

$$
\frac{(a-b)^2}{a^2 + b^2} + \frac{(a-c)^2}{c^2 + a^2} \geq \frac{(b-c)^2}{(b+c)^2}.
$$

Thus, it suffices to show that

$$
\left[ k - 1 + \frac{(b-c)^2}{(b+c)^2} \right] \left( k + \frac{2bc}{b^2 + c^2} \right) \geq k^2 - 1,
$$
which is equivalent to the obvious inequality 

\[(b - c)^4 + 2(3 - k)bc(b - c)^2 \geq 0.\]

The equality holds for \(a = b = c\).

\[\square\]

**P 1.150.** If \(a, b, c\) are non-zero and distinct real numbers, then

\[
\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + 3\left[\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2}\right] \geq 4 \left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right).
\]

**Solution.** Write the inequality as

\[
\left(\sum \frac{1}{a^2} - \sum \frac{1}{bc}\right) + 3 \sum \frac{1}{(b-c)^2} \geq 3 \sum \frac{1}{bc}.
\]

In virtue of the AM-GM inequality, it suffices to prove that

\[
2\sqrt{3\left(\sum \frac{1}{a^2} - \sum \frac{1}{bc}\right)\sum \frac{1}{(b-c)^2}} \geq 3 \sum \frac{1}{bc},
\]

which is true if

\[
4\left(\sum \frac{1}{a^2} - \sum \frac{1}{bc}\right)\sum \frac{1}{(b-c)^2} \geq 3\left(\sum \frac{1}{bc}\right)^2.
\]

Rewrite this inequality as

\[
4\left(\sum a^2b^2 - abc\sum a\right)(\sum a^2 - \sum ab)^2 \geq 3(a + b + c)^2(a - b)^2(b - c)^2(c - a)^2.
\]

Using the notations

\[p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,
\]

and the identity

\[(a - b)^2(b - c)^2(c - a)^2 = -27r^2 - 2(2p^2 - 9q)pr + p^2q^2 - 4q^3,
\]

we can write the inequality as

\[4(q^2 - 3pr)(p^2 - 3q)^2 \geq 3p^2(-27r^2 - 2(2p^2 - 9q)pr + p^2q^2 - 4q^3),
\]

which is equivalent to

\[(9pr + p^2q - 6q^2)^2 \geq 0.
\]

\[\square\]
P 1.151. Let \( a, b, c \) be positive real numbers, and let
\[
A = \frac{a}{b} + \frac{b}{a} + k, \quad B = \frac{b}{c} + \frac{c}{b} + k, \quad C = \frac{c}{a} + \frac{a}{c} + k,
\]
where \(-2 < k \leq 4\). Prove that
\[
\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq \frac{1}{k+2} + \frac{4}{A+B+C-(k+2)}.
\]

(Vasile Cîrtoaje, 2009)

Solution. Let us denote
\[ x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a}. \]
We need to show that
\[
\sum \frac{x}{x^2 + kx + 1} \leq \frac{1}{k+2} + \frac{4}{\sum x + \sum xy + 2k-2}
\]
for all positive real numbers \( x, y, z \) satisfying \( xyz = 1 \). Write this inequality as follows:
\[
\sum \left( \frac{1}{k+2} - \frac{x}{x^2 + kx + 1} \right) \geq \frac{2}{k+2} - \frac{4}{\sum x + \sum xy + 2k-2},
\]
\[
\sum \frac{(x-1)^2}{x^2 + kx + 1} \geq \frac{2\sum yz(x-1)^2}{\sum x + \sum xy + 2k-2},
\]
\[
\sum \frac{(x-1)^2[-x+y+z+x(y+z)-yz-2]}{x^2 + kx + 1} \geq 0.
\]
Since
\[
-(x+y+z+x(y+z)-yz-2) = (x+1)(y+z)-(x+yz+2)
\]
\[
= (x+1)(y+z) - (x+1)(yz+1) = -(x+1)(y-1)(z-1),
\]
the inequality is equivalent to
\[
-(x-1)(y-1)(z-1) \sum \frac{x^2-1}{x^2 + kx + 1} \geq 0,
\]
or \( E \geq 0 \), where
\[
E = -(x-1)(y-1)(z-1) \sum (x^2-1)(y^2+ky+1)(z^2+kz+1).
\]
We have
\[
\sum (x^2-1)(y^2+ky+1)(z^2+kz+1) = k(2-k) \left( \sum xy - \sum x \right) + \left( \sum x^2y^2 - \sum x^2 \right)
\]
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\[-k(2 - k)(x - 1)(y - 1)(z - 1) - (x^2 - 1)(y^2 - 1)(z^2 - 1)\]
\[= -(x - 1)(y - 1)(z - 1)[(x + 1)(y + 1)(z + 1) + k(2 - k)],\]
and hence
\[E = (x - 1)^2(y - 1)^2(z - 1)^2[(x + 1)(y + 1)(z + 1) + k(2 - k)] \geq 0,
because
\[(x + 1)(y + 1)(z + 1) + k(2 - k) \geq (2\sqrt{x})(2\sqrt{y})(2\sqrt{z}) + k(2 - k) = (2 + k)(4 - k) \geq 0.
The equality holds for \(a = b,\) or \(b = c,\) or \(c = a.\)

\[P 1.152.\] If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} \geq \frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab}.
\]
(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as follows:
\[
\sum \left( \frac{1}{b^2 + bc + c^2} - \frac{1}{2a^2 + bc} \right) \geq 0,
\]
\[
\sum \left( \frac{(a^2 - b^2) + (a^2 - c^2)}{(b^2 + bc + c^2)(2a^2 + bc)} \right) \geq 0,
\]
\[
\sum \frac{a^2 - b^2}{(b^2 + bc + c^2)(2a^2 + bc)} + \sum \frac{b^2 - a^2}{(2b^2 + ca)(c^2 + ca + a^2)} \geq 0,
\]
\[
(a^2 + b^2 + c^2 - ab - bc - ca) \sum \frac{c(a^2 - b^2)(a - b)}{(b^2 + bc + c^2)(c^2 + ca + a^2)(2a^2 + bc)(2b^2 + ca)} \geq 0.
\]
Clearly, the last form is obvious. The equality holds for \(a = b = c.\)

\[P 1.153.\] If \(a, b, c\) are nonnegative real numbers such that \(a + b + c = 3,\) then
\[
\frac{1}{2ab + 1} + \frac{1}{2bc + 1} + \frac{1}{2ca + 1} \geq \frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2}.
\]
(Vasile Cîrtoaje, 2014)
Solution. Write the inequality as

\[
\sum \frac{1}{2ab+1} - \frac{3}{2} \geq \sum \left( \frac{1}{a^2 + 2} - \frac{1}{2} \right),
\]

\[
\sum \frac{1}{2ab+1} + \sum \frac{a^2}{2(a^2 + 2)} \geq \frac{3}{2}.
\]

Let us denote

\[ q = ab + bc + ca, \quad q \leq 3. \]

By the Cauchy-Schwarz inequality, we have

\[
\sum \frac{1}{2ab+1} \geq \frac{9}{\sum (2ab + 1)} = \frac{9}{2q + 3}
\]

and

\[
\sum \frac{a^2}{2(a^2 + 2)} \geq \frac{(\sum a)^2}{\sum 2(a^2 + 2)} = \frac{9}{2(15 - 2q)}.
\]

Therefore, it suffices to prove that

\[
\frac{9}{2q + 3} + \frac{9}{2(15 - 2q)} \geq \frac{3}{2}.
\]

This inequality is true because it reduces to the obvious inequality

\[(3 - q)(9 - 2q) \geq 0.\]

The equality holds for \(a = b = c = 1\).

\[ \square \]

**P 1.154.** If \(a, b, c\) are nonnegative real numbers such that \(a + b + c = 4\), then

\[
\frac{1}{ab + 2} + \frac{1}{bc + 2} + \frac{1}{ca + 2} \geq \frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2}.
\]

(Vasile Cirtoaje, 2014)

**First Solution (by Nguyen Van Quy).** Rewrite the inequality as follows:

\[
\sum \left( \frac{2}{ab + 2} - \frac{1}{a^2 + 2} - \frac{1}{b^2 + 2} \right) \geq 0,
\]

\[
\sum \left[ \frac{a(a - b)}{(ab + 2)(a^2 + 2)} + \frac{b(b - a)}{(ab + 2)(b^2 + 2)} \right] \geq 0,
\]
\[ \sum \frac{(2 - ab)(a - b)^2(c^2 + 2)}{ab + 2} \geq 0. \]

Without loss of generality, assume that \( a \geq b \geq c \geq 0 \). Then,

\[ bc \leq ac \leq \frac{a(b + c)}{2} \leq \frac{(a + b + c)^2}{8} = 2 \]

and

\[ \sum \frac{(2 - ab)(a - b)^2(c^2 + 2)}{ab + 2} \geq \frac{(2 - ab)(a - b)^2(c^2 + 2)}{ab + 2} + \frac{(2 - ac)(a - c)(b^2 + 2)}{ac + 2} \]
\[ \geq \frac{(2 - ab)(a - b)^2(c^2 + 2)}{ab + 2} + \frac{(2 - ac)(a - b)(c^2 + 2)}{ab + 2} \]
\[ = \frac{(4 - ab - ac)(a - b)^2(c^2 + 2)}{ab + 2} \geq 0. \]

The equality holds for \( a = b = c = 4/3 \), and also for \( a = 2 \) and \( b = c = 1 \) (or any cyclic permutation).

**Second Solution.** Write the inequality as

\[ \sum \frac{1}{bc + 2} - \frac{3}{2} \geq \sum \left( \frac{1}{a^2 + 2} - \frac{1}{2} \right), \]
\[ \sum \frac{1}{bc + 2} + \sum \frac{a^2}{2(a^2 + 2)} \geq \frac{3}{2}. \]

Assume that \( a \geq b \geq c \) and denote

\[ s = \frac{b + c}{2}, \quad p = bc, \quad 0 \leq s \leq \frac{4}{3}, \quad 0 \leq p \leq s^2. \]

By the Cauchy-Schwarz inequality, we have

\[ \frac{b^2}{2(b^2 + 2)} + \frac{c^2}{2(c^2 + 2)} \geq \frac{(b + c)^2}{2(b^2 + 2) + 2(c^2 + 2) + 4} = \frac{s^2}{2s^2 - p + 2}. \]

In addition,

\[ \frac{1}{ca + 2} + \frac{1}{ab + 2} = \frac{a(b + c) + 4}{(ab + 2)(ac + 2)} = \frac{2as + 4}{a^2p + 4as + 4}. \]

Therefore, it suffices to show that \( E(a, b, c) \geq 0 \), where

\[ E(a, b, c) = \frac{1}{p + 2} + \frac{2(as + 2)}{a^2p + 4as + 4} + \frac{a^2}{2(a^2 + 2)} + \frac{s^2}{2s^2 - p + 2} - \frac{3}{2}. \]

We will prove that

\[ E(a, b, c) \geq E(a, s, s) \geq 0. \]
We have
\[
E(a, b, c) - E(a, s, s) = \left(\frac{1}{p+2} - \frac{1}{s^2 + 2}\right) + 2(as + 2)\left(\frac{1}{a^2p + 4as + 4} - \frac{1}{a^2s^2 + 4as + 4}\right)
\]
\[+ s^2\left(\frac{1}{2s^2 - p + 2} - \frac{1}{s^2 + 2}\right)\]
\[= \frac{s^2 - p}{(p+2)(s^2 + 2)} + \frac{2a^2(as + 2)(s^2 - p)}{(a^2p + 4as + 4)(a^2s^2 + 4as + 4)}\]
\[\quad - \frac{2a^2(2s^2 - p + 2)}{(s^2 + 2)(2s^2 - p + 2)}.\]

Since \(s^2 - p \geq 0\), it remains to show that
\[
\frac{1}{(p+2)(s^2 + 2)} + \frac{2a^2(as + 2)}{(a^2p + 4as + 4)(a^2s^2 + 4as + 4)} \geq \frac{s^2}{(s^2 + 2)(2s^2 - p + 2)},
\]
which is equivalent to
\[
\frac{2a^2(as + 2)}{(a^2p + 4as + 4)(a^2s^2 + 4as + 4)} \geq \frac{p(s^2 + 1) - 2}{(p+2)(s^2 + 2)(2s^2 - p + 2)}.
\]

Since
\[a^2p + 4as + 4 \leq a^2s^2 + 4as + 4 = (as + 2)^2\]
and
\[2s^2 - p + 2 \geq s^2 + 2,\]
it is enough to prove that
\[
\frac{2a^2}{(as + 2)^3} \geq \frac{p(s^2 + 1) - 2}{(p+2)(s^2 + 2)^2}.
\]

In addition, since
\[as + 2 = (4 - 2s)s + 2 \leq 4\]
and
\[
p(s^2 + 1) - 2 = s^2 + 1 - \frac{2(s^2 + 2)}{p + 2} \leq s^2 + 1 - \frac{2(s^2 + 2)}{s^2 + 2} = s^2 - 1,
\]
it suffices to show that
\[
\frac{a^2}{32} \geq \frac{s^2 - 1}{(s^2 + 2)^2},
\]
which is equivalent to
\[\quad (2-s)^2(2+s)^2 \geq 8(s^2 - 1).\]
Indeed, for the non-trivial case $1 < s \leq \frac{4}{3}$, we have

$$(2-s)^2(2+s)^2 - 8(s^2-1) \geq \left(2 - \frac{4}{3}\right)^2 (2+s)^2 - 8(s^2-1) = \frac{4}{9}(s^4 - 14s^2 + 22)$$

$$= \frac{4}{9}[(7-s)^2 - 27] \geq \frac{4}{9}\left[\left(\frac{7}{9}\right)^2 - 27\right] = \frac{88}{729} > 0.$$

To end the proof, we need to show that $E(a, s, s) \geq 0$. Notice that $E(a, s, s)$ can be found from $E(a, b, c)$ by replacing $p$ with $s^2$. We get

$$E(a, s, s) = \frac{1}{s^2+2} + \frac{2}{as+2} + \frac{a^2}{2(a^2+2)} + \frac{s^2}{s^2+2} - \frac{3}{2}$$

$$= \frac{2(s^2+2)(1+2s-s^2)(2s^2-8s+9)}{(s-1)^2(3s-4)^2} \geq 0.$$

$\square$

**P 1.155.** If $a, b, c$ are nonnegative real numbers, no two of which are zero, then

(a) $$\frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} \leq 1;$$

(b) $$\frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2)} \leq 1.$$

*(Vasile Cîrtoaje, 2014)*

**Solution.** (a) **First Solution.** Consider the non-trivial case where $a, b, c$ are distinct and write the inequality as follows:

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} \leq \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2(a^2 + b^2 + c^2)}.$$  

$$\frac{(a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2)}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} \leq \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a-b)^2(b-c)^2(c-a)^2},$$

$$\sum \frac{1}{(b^2 + c^2)(c^2 + a^2)} \leq \sum \frac{1}{(b-c)^2(c-a)^2}.$$

Since

$$a^2 + b^2 \geq (a-b)^2, \quad b^2 + c^2 \geq (b-c)^2, \quad c^2 + a^2 \geq (c-a)^2,$$

the conclusion follows. The equality holds for $a = b = c$. 
**Second Solution.** Assume that \( a \geq b \geq c \). We have

\[
\frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{(a - b)^2(b - c)^2(c - a)^2}{(a^2 + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2)} \leq \frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{(a - b)^2(a - c)^2}{(a^2 + b^2)(a^2 + c^2)} \leq \frac{2ab + c^2}{a^2 + b^2 + c^2} + \frac{(a - b)^2a^2}{a^2(a^2 + b^2 + c^2)} = \frac{2ab + c^2 + (a - b)^2}{a^2 + b^2 + c^2} = 1.
\]

(b) Consider the non-trivial case where \( a, b, c \) are distinct and write the inequality as follows:

\[
\frac{(a - b)^2(b - c)^2(c - a)^2}{(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2)} \leq \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2(a^2 + b^2 + c^2)},
\]

\[
\frac{2(a^2 + b^2 + c^2)}{(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2)} \leq \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{(a - b)^2(b - c)^2(c - a)^2},
\]

\[
\sum \frac{1}{(a - b)^2(a - c)^2} \geq \frac{2(a^2 + b^2 + c^2)}{(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2)}.
\]

Assume that \( a = \min\{a, b, c\} \) and use the substitution

\[ b = a + x, \quad c = a + y, \quad x, y \geq 0. \]

The inequality can be written as

\[
\frac{1}{x^2y^2} + \frac{1}{x^2(x - y)^2} + \frac{1}{y^2(x - y)^2} \geq 2f(a),
\]

where

\[
f(a) = \frac{3a^2 + 2(x + y)a + x^2 + y^2}{(a^2 + xa + x^2)(a^2 + ya + y^2)[a^2 + (x + y)a + x^2 - xy + y^2]}.\]

We will show that

\[
\frac{1}{x^2y^2} + \frac{1}{x^2(x - y)^2} + \frac{1}{y^2(x - y)^2} \geq 2f(0) \geq 2f(a).
\]

We have

\[
\frac{1}{x^2y^2} + \frac{1}{x^2(x - y)^2} + \frac{1}{y^2(x - y)^2} - 2f(0) = \frac{2(x^2 + y^2 - xy)}{x^2y^2(x - y)^2} - \frac{2(x^2 + y^2)}{x^2y^2(x^2 - xy + y^2)} = \frac{2}{(x - y)^2(x^2 - xy + y^2)} \geq 0.
\]
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Also, since
\[(a^2 + xa + x^2)(a^2 + ya + y^2) \geq (x^2 + y^2)a^2 + xy(x + y)a + x^2y^2\]
and
\[a^2 + (x + y)a + x^2 - xy + y^2 \geq x^2 - xy + y^2,\]
we get \(f(a) \leq g(a)\), where
\[g(a) = \frac{3a^2 + 2(x + y)a + x^2 + y^2}{[(x^2 + y^2)a^2 + xy(x + y)a + x^2y^2](x^2 - xy + y^2)}.\]

Therefore,
\[f(0) - f(a) \geq \frac{x^2 + y^2}{x^2y^2(x^2 - xy + y^2)} - g(a) = \frac{(x^4 - x^2y^2 + y^4)a^2 + xy(x + y)(x - y)^2a}{x^2y^2(x^2 - xy + y^2)[(x^2 + y^2)a^2 + xy(x + y)a + x^2y^2]} \geq 0.\]

Thus, the proof is completed. The equality holds for \(a = b = c\).

\[\Box\]

**P 1.156.** If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{45}{8(a^2 + b^2 + c^2) + 2(ab + bc + ca)}.
\]

(Vasile Cîrtoaje, 2014)

**First Solution** (by Nguyen Van Quy). Multiplying by \(a^2 + b^2 + c^2\), the inequality becomes
\[
\sum \frac{a^2}{b^2 + c^2} + 3 \geq \frac{45(a^2 + b^2 + c^2)}{8(a^2 + b^2 + c^2) + 2(ab + bc + ca)}.
\]

Applying the Cauchy-Schwarz inequality, we have
\[
\sum \frac{a^2}{b^2 + c^2} \geq \frac{(\sum a^2)^2}{\sum a^2(b^2 + c^2)} = \frac{(a^2 + b^2 + c^2)^2}{2(a^2b^2 + b^2c^2 + c^2a^2)}.
\]

Therefore, it suffices to show that
\[
\frac{(a^2 + b^2 + c^2)^2}{2(a^2b^2 + b^2c^2 + c^2a^2)} + 3 \geq \frac{45(a^2 + b^2 + c^2)}{8(a^2 + b^2 + c^2) + 2(ab + bc + ca)},
\]
which is equivalent to
\[
\frac{(a^2 + b^2 + c^2)^2}{a^2b^2 + b^2c^2 + c^2a^2} - 3 \geq \frac{45(a^2 + b^2 + c^2)}{4(a^2 + b^2 + c^2) + ab + bc + ca} - 9,
\]
\[
\frac{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}{a^2b^2 + b^2c^2 + c^2a^2} \geq \frac{9(a^2 + b^2 + c^2 - ab - bc - ca)}{4(a^2 + b^2 + c^2) + ab + bc + ca}.
\]

By Schur's inequality of degree four, we have
\[
a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq (a^2 + b^2 + c^2 - ab - bc - ca)(ab + bc + ca) \geq 0.
\]

Therefore, it suffices to show that
\[
[4(a^2 + b^2 + c^2) + ab + bc + ca](ab + bc + ca) \geq 9(a^2b^2 + b^2c^2 + c^2a^2).
\]

Since
\[
(ab + bc + ca)^2 \geq a^2b^2 + b^2c^2 + c^2a^2,
\]
this inequality is true if
\[
4(a^2 + b^2 + c^2)(ab + bc + ca) \geq 8(a^2b^2 + b^2c^2 + c^2a^2),
\]
which is equivalent to the obvious inequality
\[
ab(a - b)^2 + bc(b - c)^2 + ca(c - a)^2 + abc(a + b + c) \geq 0.
\]
The equality holds for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation).

**Second Solution.** Write the inequality as \(f_6(a, b, c) \geq 0\), where
\[
f_6(a, b, c) = [8(a^2 + b^2 + c^2) + 2(ab + bc + ca)]\sum(a^2 + b^2)(a^2 + c^2) - 45 \prod(b^2 + c^2).
\]

Clearly, \(f_6(a, b, c)\) has the same highest coefficient \(A\) as
\[
f(a, b, c) = -45 \prod(b^2 + c^2) = -45 \prod(p^2 - 2q - a^2),
\]
where \(p = a + b + c\) and \(q = ab + bc + ca\); that is,
\[
A = 45.
\]
Since \(A > 0\), we will apply the highest coefficient cancellation method. We have
\[
f_6(a, 1, 1) = 4a(2a + 5)(a^2 + 1)(a - 1)^2,
\]
\[
f_6(0, b, c) = (b - c)^2[8(b^4 + c^4) + 18bc(b^2 + c^2) + 15b^2c^2].
\]
Since 
\[ f_6(1, 1, 1) = f_6(0, 1, 1) = 0, \]
define the homogeneous function 
\[ P(a, b, c) = abc + B(a + b + c)^3 + C(a + b + c)(ab + bc + ca) \]
such that \( P(1, 1, 1) = P(0, 1, 1) = 0; \) that is, 
\[ P(a, b, c) = abc + \frac{1}{9}(a + b + c)^3 - \frac{4}{9}(a + b + c)(ab + bc + ca). \]
We will show that the following sharper inequality holds 
\[ f_6(a, b, c) \geq 45P^2(a, b, c). \]
Let us denote 
\[ g_6(a, b, c) = f_6(a, b, c) - 45P^2(a, b, c). \]
Clearly, \( g_6(a, b, c) \) has the highest coefficient \( A = 0. \) By P 2.76-(a) in Volume 1, it suffices to prove that \( g_6(a, 1, 1) \geq 0 \) and \( g_6(0, b, c) \geq 0 \) for all \( a, b, c \geq 0. \) We have 
\[ P(a, 1, 1) = \frac{a(a - 1)^2}{9}, \]
\[ g_6(a, 1, 1) = f_6(a, 1, 1) - 45P^2(a, 1, 1) = \frac{a(a - 1)^2(67a^3 + 190a^2 + 67a + 180)}{9} \geq 0. \]
Also, we have 
\[ P(0, b, c) = \frac{(b + c)(b - c)^2}{9}, \]
\[ g_6(0, b, c) = f_6(0, b, c) - 45P^2(0, b, c) \]
\[ = \frac{(b - c)^2[67(b^4 + c^4) + 162bc(b^2 + c^2) + 145b^2c^2]}{9} \geq 0. \]
\[ \square \]

P 1.157. If \( a, b, c \) are real numbers, no two of which are zero, then 
\[ \frac{a^2 - 7bc}{b^2 + c^2} + \frac{b^2 - 7ca}{a^2 + b^2} + \frac{c^2 - 7ab}{a^2 + b^2} + \frac{9(ab + bc + ca)}{a^2 + b^2 + c^2} \geq 0. \]

(Vasile Cîrtoaje, 2014)
Solution. Let

\[ p = a + b + c, \quad q = ab + bc + ca, \quad r = abc. \]

Write the inequality as \( f_8(a, b, c) \geq 0 \), where

\[
f_8(a, b, c) = (a^2 + b^2 + c^2) \sum (a^2 - 7bc)(a^2 + b^2)(a^2 + c^2)
+ 9(ab + bc + ca) \prod (b^2 + c^2)
\]

is a symmetric homogeneous polynomial of degree eight. Always, \( f_8(a, b, c) \) can be written in the form

\[
f_8(a, b, c) = A(p, q)r^2 + B(p, q)r + C(p, q),
\]

where the highest polynomial \( A(p, q) \) has the form \( \alpha p^2 + \beta q \). Since

\[
f_8(a, b, c) = (p^2 - 2q) \sum (a^2 - 7bc)(p^2 - 2q - c^2)(p^2 - 2q - b^2)
+ 9q \prod (p^2 - 2q - a^2),
\]

\( f_8(a, b, c) \) has the same highest polynomial as

\[
g_8(a, b, c) = (p^2 - 2q) \sum (a^2 - 7bc)b^2c^2 + 9q(-a^2b^2c^2)
= (p^2 - 2q)(3r^2 - 7 \sum b^3c^3) - 9qr^2;
\]

that is,

\[
A(p, q) = (p^2 - 2q)(3r^2 - 3) - 9q = -9(p^2 - 3q).
\]

Since \( A(p, q) \leq 0 \) for all real \( a, b, c \), by Lemma below, it suffices to prove that \( f_8(a, 1, 1) \geq 0 \) for all \( a, b, c \geq 0 \). We have

\[
f_8(a, 1, 1) = (a^2 + 1)(a - 1)^2(a + 2)^2(a^2 - 2a + 3) \geq 0.
\]

The equality holds for \( a = b = c \), and also for \( -a/2 = b = c \) (or any cyclic permutation).

Lemma. Let

\[ p = a + b + c, \quad q = ab + bc + ca, \quad r = abc, \]

and let \( f_8(a, b, c) \) be a symmetric homogeneous polynomial of degree eight written in the form

\[
f_8(a, b, c) = A(p, q)r^2 + B(p, q)r + C(p, q),
\]

where \( A(p, q) \leq 0 \) for all real \( a, b, c \). The inequality \( f_8(a, b, c) \geq 0 \) holds for all real numbers \( a, b, c \) if and only if \( f_8(a, 1, 1) \geq 0 \) for all real \( a \).

Proof. For fixed \( p \) and \( q \),

\[
h_8(r) = A(p, q)r^2 + B(p, q)r + C(p, q)
\]

is the highest polynomial of \( f_8(a, b, c) \), so that

\[
h_8(r) = \alpha r^2 + \beta r + \gamma
\]

is the highest polynomial of \( f_8(a, b, c) \) and it is non-negative (for real \( r \) and \( \alpha \geq 0, \beta \geq 0, \gamma \leq 0 \)).
Symmetric Rational Inequalities

is a concave quadratic function of \( r \). Therefore, \( h_8(r) \) is minimal when \( r \) is minimal or maximal; this is, according to P 1.53 in Volume 1, when \( f_8(a, 1, 1) \geq 0 \) and \( f_8(a, 0, 0) \geq 0 \) for all real \( a \). Notice that the condition "\( f_8(a, 0, 0) \geq 0 \) for all real \( a \)" is not necessary because it follows from the condition "\( f_8(a, 1, 1) \geq 0 \) for all real \( a \)" as follows:

\[
f_8(a, 0, 0) = \lim_{t \to 0} f_8(a, t, t) = \lim_{t \to 0} t^8 f_8(a/t, 1, 1) \geq 0.
\]

Notice that \( A(p, q) \) is called the highest polynomial of \( f_8(a, b, c) \).

**Remark.** This Lemma can be extended as follow.

- The inequality \( f_8(a, b, c) \geq 0 \) holds for all real numbers \( a, b, c \) satisfying \( A(p, q) \leq 0 \) if and only if \( f_8(a, 1, 1) \geq 0 \) for all real \( a \) such that \( A(a + 2, 2a + 1) \leq 0 \).

\[ \square \]

**P 1.158.** If \( a, b, c \) are real numbers such that \( abc \neq 0 \), then

\[
\frac{(b + c)^2}{a^2} + \frac{(c + a)^2}{b^2} + \frac{(a + b)^2}{c^2} \geq 2 + \frac{10(a + b + c)^2}{3(a^2 + b^2 + c^2)}.
\]

(Vasile Cîrtoaje and Michael Rozenberg, 2014)

**Solution.** Let

\[
p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.
\]

Write the inequality as \( f_8(a, b, c) \geq 0 \), where

\[
f_8(a, b, c) = 3(a^2 + b^2 + c^2) \left[ \sum b^2 c^2 (b + c)^2 - 2a^2 b^2 c^2 \right] - 10a^2 b^2 c^2 (a + b + c)^2.
\]

From

\[
\sum b^2 c^2 (b + c)^2 - 2a^2 b^2 c^2 = \sum b^2 c^2 (p - a)^2 - 2r^2 = p^2 \sum b^2 c^2 - 2pqr + r^2,
\]

it follows that \( f_8(a, b, c) \) has the same highest polynomial as

\[
3(a^2 + b^2 + c^2)r^2 - 10r^2(a + b + c)^2;
\]

that is,

\[
A(p, q) = 3(p^2 - 2q) - 10p^2 = -7p^2 - 6q.
\]

There are two cases to consider.

**Case 1:** \( A(p, q) \leq 0 \). According to Remark from the preceding P 1.157, it suffices to show that \( f_8(a, 1, 1) \geq 0 \) for all real \( a \) such that \( A(a + 2, 2a + 1) \leq 0 \). Indeed, we have

\[
f_8(a, 1, 1) = 3(a^2 + 2)[4 + 2a^2(a + 1)^2 - 2a^2] - 10a^2(a + 2)^2
\]

\[
= 2(3a^6 + 6a^5 + a^4 - 8a^3 - 14a^2 + 12)
\]

\[
= 2(a - 1)^2(3a^4 + 12a^3 + 22a^2 + 24a + 12)
\]

\[
= 2(a - 1)^2[3(a + 1)^4 + (2a + 3)^2] \geq 0.
\]
Case 2: \( A(p, q) > 0 \). We will show that there exist two real numbers \( B \) and \( C \) such that the following sharper inequality holds:

\[
f_\delta(a, b, c) \geq A(p, q)P^2(a, b, c),
\]

where

\[
P(a, b, c) = r + Bp^3 + Cpq.
\]

Let us denote

\[
g_\delta(a, b, c) = f_\delta(a, b, c) - A(p, q)P^2(a, b, c).
\]

We see that the highest polynomial of \( g_\delta(a, b, c) \) is zero. Thus, according to Remark from P 1.157, it suffices to prove that \( g(a) \geq 0 \) for all real \( a \), where \( g(a) = g_\delta(a, 1, 1) \). We have

\[
g(a) = f_\delta(a, 1, 1) - A(a + 2, 2a + 1)P^2(a, 1, 1),
\]

where

\[
f_\delta(a, 1, 1) = 2(3a^6 + 6a^5 + a^4 - 8a^3 - 14a^2 + 12),
\]

\[
A(a + 2, 2a + 1) = -7(a + 2)^2 - 6(2a + 1) = -7a^2 - 40a - 34,
\]

\[
P(a, 1, 1) = a + B(a + 2)^3 + C(a + 2)(2a + 1).
\]

Since

\[
g(-2) = 72 - 18(-2)^2 = 0,
\]

a necessary condition to have \( g(a) \geq 0 \) in the vicinity of \(-2\) is \( g'(-2) = 0 \). This condition involves \( C = -1 \). We can check that \( g''(-2) = 0 \) for this value of \( C \). Thus, a necessary condition to have \( g(a) \geq 0 \) in the vicinity of \(-2\) is \( g'''(-2) = 0 \) is necessary. This condition involves \( B = 2/3 \). For these values of \( B \) and \( C \), we have

\[
g(a) = f_\delta(a, 1, 1) + (7a^2 + 40a + 34)\left[a + \frac{2}{3}(a + 2)^3 - (a + 2)(2a + 1)\right]^2
\]

\[
= \frac{2}{9}(a + 2)^4(14a^4 + 52a^3 + 117a^2 + 154a + 113).
\]

Since

\[
14a^4 + 52a^3 + 117a^2 + 154a + 113 = (a^2 + 1)^2 + 13a^2(a + 2)^2 + 7(9a^2 + 22a + 16) > 0,
\]

the proof is completed. The equality holds for \( a = b = c \).

\[\square\]

P 1.159. If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then

\[
\frac{a^2 - 4bc}{b^2 + c^2} + \frac{b^2 - 4ca}{a^2 + b^2} + \frac{c^2 - 4ab}{a^2 + b^2} + \frac{9(ab + bc + ca)}{a^2 + b^2 + c^2} \geq \frac{9}{2}.
\]

(Vasile Cirtoaje, 2014)
Solution. Let
\[ p = a + b + c, \quad q = ab + bc + ca, \quad r = abc. \]
Write the inequality as \( f_8(a, b, c) \geq 0 \), where
\[
f_8(a, b, c) = 2(a^2 + b^2 + c^2) \sum (a^2 - 4bc)(a^2 + b^2)(a^2 + c^2) \\
+ 9(2ab + 2bc + 2ca - a^2 - b^2 - c^2) \prod (b^2 + c^2)
\]
is a symmetric homogeneous polynomial of degree eight. Always, \( f_8(a, b, c) \) can be written in the form
\[
f_8(a, b, c) = A(p, q)r^2 + B(p, q)r + C(p, q),
\]
where \( A(p, q) = \alpha p^2 + \beta q \) is called the highest polynomial of \( f_8(a, b, c) \). Since
\[
f_8(a, b, c) = 2(p^2 - 2q) \sum (a^2 - 4bc)(p^2 - 2q - c^2)(p^2 - 2q - b^2) \\
+ 9(4q - p^2) \prod (p^2 - 2q - a^2),
\]
\( f_8(a, b, c) \) has the same highest polynomial as
\[
g_8(a, b, c) = 2(p^2 - 2q) \sum (a^2 - 4bc)b^2c^2 + 9(4q - p^2)(-a^2b^2c^2) \\
= 2(p^2 - 2q)(3r^2 - 4 \sum b^3c^3) - 9(4q - p^2)r^2;
\]
that is,
\[
A(p, q) = 2(p^2 - 2q)(3 - 12) - 9(4q - p^2) = -9p^2.
\]
Since \( A(p, q) \leq 0 \) for all \( a, b, c \geq 0 \), by Lemma below, it suffices to prove that \( f_8(a, 1, 1) \geq 0 \) and \( f_8(0, b, c) \geq 0 \) for all \( a, b, c \geq 0 \). We have
\[
f_8(a, 1, 1) = 2a(a + 4)(a^2 + 1)(a - 1)^4 \geq 0
\]
and
\[
f_8(0, b, c) = (b^2 + c^2)(2E - 9F),
\]
where
\[
E = -4b^3c^3 + (b^2 + c^2)(b^4 + c^4), \quad F = b^2c^2(b - c)^2.
\]
Since
\[
E \geq -4b^3c^3 + 2bc(b^4 + c^4) = 2bc(b^2 - c^2)^2,
\]
we have
\[
2E - 9F \geq 4bc(b^2 - c^2)^2 - 9b^2c^2(b - c)^2 = bc(b - c)^2[4(b + c)^2 - 9bc] \geq 0.
\]
Thus, the proof is completed. The equality holds for \( a = b = c \), and also for \( a = 0 \) and \( b = c \) (or any cyclic permutation).
Lemma. Let

\[ p = a + b + c, \quad q = ab + bc + ca, \quad r = abc, \]

and let \( f_8(a, b, c) \) be a symmetric homogeneous polynomial of degree eight written in the form

\[ f_8(a, b, c) = A(p, q)r^2 + B(p, q)r + C(p, q), \]

where \( A(p, q) \leq 0 \) for all \( a, b, c \geq 0 \). The inequality \( f_8(a, b, c) \geq 0 \) holds for all nonnegative real numbers \( a, b, c \) if and only if \( f_8(a, 1, 1) \geq 0 \) and \( f_8(0, b, c) \geq 0 \) for all \( a, b, c \geq 0 \).

Proof. For fixed \( p \) and \( q \),

\[ h_8(r) = A(p, q)r^2 + B(p, q)r + C(p, q) \]

is a concave quadratic function of \( r \). Therefore, \( h_8(r) \) is minimal when \( r \) is minimal or maximal. This is, according to P 2.57 in Volume 1, when \( b = c \) or \( a = 0 \). Thus, the conclusion follows. Notice that \( A(p, q) \) is called the highest polynomial of \( f_8(a, b, c) \).

Remark. This Lemma can be extended as follows.

- The inequality \( f_8(a, b, c) \geq 0 \) holds for all \( a, b, c \geq 0 \) satisfying \( A(p, q) \leq 0 \) if and only if \( f_8(a, 1, 1) \geq 0 \) and \( f_8(0, b, c) \geq 0 \) for all \( a, b, c \geq 0 \) such that \( A(a + 2, 2a + 1) \leq 0 \) and \( A(b + c, bc) \leq 0 \).

\( \square \)

P 1.160. If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then

\[ \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq 1 + \frac{9(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2}. \]

(Vasile Cîrtoaje, 2014)

Solution. Consider the non-trivial case where \( a, b, c \) are distinct and \( a = \min\{a, b, c\} \).

Write the inequality as follows:

\[ \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2(ab + bc + ca)} \geq \frac{9(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2}, \]

\[ \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a-b)^2(b-c)^2(c-a)^2} \geq \frac{18(ab + bc + ca)}{(a+b)^2(b+c)^2(c+a)^2}, \]

\[ \sum \frac{1}{(b-a)^2(c-a)^2} \geq \frac{18(ab + bc + ca)}{(a+b)^2(a+c)^2(b+c)^2}. \]

Since

\[ \sum \frac{1}{(b-a)^2(c-a)^2} \geq \frac{1}{b^2c^2} + \frac{1}{b^2(b-c)^2} + \frac{1}{c^2(b-c)^2} = \frac{2(b^2 + c^2 - bc)}{b^2c^2(b-c)^2} \]
and
\[ \frac{ab + bc + ca}{(a + b)^2(a + c)^2(b + c)^2} \leq \frac{ab + bc + ca}{(ab + bc + ca)^2(b + c)^2} \leq \frac{1}{bc(b + c)^2}, \]
it suffices to show that
\[ \frac{b^2 + c^2 - bc}{b^2c^2(b - c)^2} \geq \frac{9}{bc(b + c)^2}. \]
Write this inequality as follows:
\[ \frac{(b + c)^2 - 3bc}{bc} \geq \frac{9(b + c)^2 - 36bc}{(b + c)^2}, \]
\[ \frac{(b + c)^2}{bc} - 12 + \frac{36bc}{(b + c)^2} \geq 0, \]
\[ (b + c)^4 - 12bc(b + c)^2 + 36b^2c^2 \geq 0, \]
\[ [(b + c)^2 - 6bc]^2 \geq 0. \]
Thus, the proof is completed. The equality holds for \( a = b = c \), and also for \( a = 0 \) and \( b/c + c/b = 4 \) (or any cyclic permutation).

\[ \square \]

**P 1.161.** If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then
\[ \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq 1 + (1 + \sqrt{2}) \frac{(a-b)(b-c)(c-a)}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}. \]
*(Vasile Cirtoaje, 2014)*

**Solution.** Consider the non-trivial case where \( a, b, c \) are distinct and denote \( k = 1 + \sqrt{2} \). Write the inequality as follows:
\[ \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2(ab + bc + ca)} \geq \frac{k^2(a-b)(b-c)(c-a)}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}, \]
\[ \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a-b)^2(b-c)^2(c-a)^2} \geq \frac{2k^2(ab + bc + ca)}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}, \]
\[ \sum \frac{1}{(b-a)^2(c-a)^2} \geq \frac{2k^2(ab + bc + ca)}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}. \]
Assume that \( a = \min\{a, b, c\} \), and use the substitution
\[ b = a + x, \quad c = a + y, \quad x, y \geq 0. \]
The inequality becomes
\[
\frac{1}{x^2y^2} + \frac{1}{x^2(x - y)^2} + \frac{1}{y^2(x - y)^2} \geq 2k^2 f(a),
\]
where
\[
f(a) = \frac{3a^2 + 2(x + y)a + xy}{(2a^2 + 2xa + x^2)(2a^2 + 2ya + y^2)[2a^2 + 2(x + y)a + x^2 + y^2]}.
\]
We will show that
\[
\frac{1}{x^2y^2} + \frac{1}{x^2(x - y)^2} + \frac{1}{y^2(x - y)^2} \geq 2k^2 f(0) \geq 2k^2 f(a).
\]
We have
\[
\begin{align*}
\frac{1}{x^2y^2} + \frac{1}{x^2(x - y)^2} + \frac{1}{y^2(x - y)^2} - 2k^2 f(0) &= \frac{2(x^2 + y^2 - xy)}{x^2y^2(x - y)^2} - \frac{2k^2xy}{x^2y^2(x + y^2)} \\
&= \frac{2(x^2 + y^2 - (2 + \sqrt{2})xy)^2}{x^2y^2(x - y)^2(x^2 - xy + y^2)} \geq 0.
\end{align*}
\]
Also, since
\[
(2a^2 + 2xa + x^2)(2a^2 + 2ya + y^2) \geq 2(x^2 + y^2)a^2 + 2xy(x + y)a + x^2y^2
\]
and
\[
2a^2 + 2(x + y)a + x^2 + y^2 \geq x^2 + y^2,
\]
we get \( f(a) \leq g(a) \), where
\[
g(a) = \frac{3a^2 + 2(x + y)a + xy}{[2(x^2 + y^2)a^2 + 2xy(x + y)a + x^2y^2](x^2 + y^2)}.
\]
Therefore,
\[
\begin{align*}
f(0) - f(a) &\geq \frac{1}{xy(x^2 + y^2)} - g(a) \\
&= \frac{(2x^2 + 2y^2 - 3xy)a^2}{xy(x^2 + y^2)[2(x^2 + y^2)a^2 + 2xy(x + y)a + x^2y^2]} \geq 0.
\end{align*}
\]
Thus, the proof is completed. The equality holds for \( a = b = c \), and also for \( a = 0 \) and \( b/c + c/b = 2 + \sqrt{2} \) (or any cyclic permutation).
**P 1.162.** If $a, b, c$ are nonnegative real numbers, no two of which are zero, then

$$\frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \geq \frac{5}{3a + b + c} + \frac{5}{3b + c + a} + \frac{5}{3c + a + b}.$$ 

**Solution.** Write the inequality as follows:

\[
\sum \left( \frac{2}{b + c} - \frac{5}{3a + b + c} \right) \geq 0,
\]

\[
\sum \frac{2a - b - c}{(b + c)(3a + b + c)} \geq 0,
\]

\[
\sum \frac{a - b}{(b + c)(3a + b + c)} + \sum \frac{a - c}{(b + c)(3a + b + c)} \geq 0,
\]

\[
\sum \frac{a - b}{(b + c)(3a + b + c)} + \sum \frac{b - a}{(c + a)(3b + c + a)} \geq 0,
\]

\[
\sum \frac{(a - b)^2(a + b - c)}{(b + c)(c + a)(3a + b + c)(3b + c + a)},
\]

\[
\sum (b - c)^2S_a \geq 0,
\]

where

\[
S_a = (b + c - a)(b + c)(3a + b + c).
\]

Assume that $a \geq b \geq c$. Since $S_c > 0$, it suffices to show that

\[(b - c)^2S_a + (a - c)^2S_b \geq 0.\]

Since $S_b \geq 0$ and $(a - c)^2 \geq (b - c)^2$, we have

\[(b - c)^2S_a + (a - c)^2S_b \geq (b - c)^2S_a + (b - c)^2S_b = (b - c)^2(S_a + S_b).\]

Thus, it is enough to prove that $S_a + S_b \geq 0$, which is equivalent to

\[(c + a - b)(c + a)(3b + c + a) \geq (b + c - a)(b + c)(3a + b + c).\]

Consider the nontrivial case where $b + c - a > 0$. Since $c + a - b \geq b + c - a$, we only need to show that

\[(c + a)(3b + c + a) \geq (b + c)(3a + b + c).\]

Indeed,

\[(c + a)(3b + c + a) - (b + c)(3a + b + c) = (a - b)(a + b - c) \geq 0.\]

Thus, the proof is completed. The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation). \qed
P 1.163. If $a, b, c$ are real numbers, no two of which are zero, then

(a) \[ \frac{8a^2 + 3bc}{b^2 + bc + c^2} + \frac{8b^2 + 3ca}{c^2 + ca + a^2} + \frac{8c^2 + 3ab}{a^2 + ab + b^2} \geq 11; \]

(b) \[ \frac{8a^2 - 5bc}{b^2 - bc + c^2} + \frac{8b^2 - 5ca}{c^2 - ca + a^2} + \frac{8c^2 - 5ab}{a^2 - ab + b^2} \geq 9. \]

*(Vasile Cirtoaje, 2011)*

**Solution.** Consider the more general inequality

\[
\frac{a^2 + mbc}{b^2 + kbc + c^2} + \frac{b^2 + mca}{c^2 + kca + a^2} + \frac{c^2 + mab}{a^2 + kab + b^2} \geq \frac{3(m+1)}{k+2}.
\]

Let $p = a + b + c$ and $q = ab + bc + ca$. Write the inequality in the form $f_6(a, b, c) \geq 0$, where

\[
f_6(a, b, c) = (k+2) \sum (a^2 + mbc)(a^2 + kab + b^2)(a^2 + kac + c^2) - 3(m+1) \prod (b^2 + kbc + c^2).
\]

From

\[
f_6(a, b, c) = (k+2) \sum (a^2 + mbc)(kab - c^2 + p^2 - 2q)(kac - b^2 + p^2 - 2q) - 3(m+1) \prod (kbc - a^2 + p^2 - 2q).
\]

it follows that $f_6(a, b, c)$ has the same highest coefficient $A$ as

\[
(k+2)P_2(a, b, c) - 3(m+1)P_3(a, b, c),
\]

where

\[
P_2(a, b, c) = \sum (a^2 + mbc)(kab - c^2)(kac - b^2),
\]

\[
P_3(a, b, c) = \prod (kbc - a^2).
\]

According to Remark 2 from P 1.75 in Volume 1,

\[
A = (k+2)P_2(1, 1, 1) - 3(m+1)P_3(1, 1, 1) = 3(k+2)(m+1)(k-1)^2 - 3(m+1)(k-1)^3 = 9(m+1)(k-1)^2.
\]

Also, we have

\[
f_6(a, 1, 1) = (k+2)(a^2 + ka + 1)(a - 1)^2[a^2 + (k+2)a + 1 + 2k - 2m].
\]

(a) For our particular case $m = 3/8$ and $k = 1$, we have $A = 0$. Therefore, according to P 1.75 in Volume 1, it suffices to prove that $f_6(a, 1, 1) \geq 0$ for all real $a$. Indeed,

\[
f_6(a, 1, 1) = 3(a^2 + a + 1)(a - 1)^2 \left( a + \frac{3}{2} \right)^2 \geq 0.
\]
Thus, the proof is completed. The equality holds for \(a = b = c\), and also for \(-2a/3 = b = c\) (or any cyclic permutation).

(b) For \(m = -5/8\) and \(k = -1\), we have \(A = 27/2\) and

\[
f_6(a, 1, 1) = \frac{1}{4}(a^2 - a + 1)(a - 1)^2(2a + 1)^2.
\]

Since \(A > 0\), we will use the highest coefficient cancellation method. Define the homogeneous polynomial

\[
P(a, b, c) = r + Bp^3 + Cpq,
\]

where \(B\) and \(C\) are real constants. Since the desired inequality becomes an equality for \(a = b = c = 1\), and also for \(a = -1\) and \(b = c = 2\), determine \(B\) and \(C\) such that

\[
P(1, 1, 1) = P(-1, 2, 2) = 0.
\]

We find

\[
B = \frac{4}{27}, \quad C = \frac{-5}{9},
\]

when

\[
P(a, 1, 1) = \frac{2}{27}(a - 1)^2(2a + 1), \quad P(a, 0, 0) = Ba^3 = \frac{4}{27}a^3.
\]

We will show that

\[
f_6(a, b, c) \geq \frac{27}{2}p^2(a, b, c).
\]

Let us denote

\[
g_6(a, b, c) = f_6(a, b, c) - \frac{27}{2}p^2(a, b, c).
\]

Since \(g_6(a, b, c)\) has the highest coefficient \(A = 0\), it suffices to prove that \(g_6(a, 1, 1) \geq 0\) for all real \(a\) (see P 1.75 in Volume 1). Indeed,

\[
g_6(a, 1, 1) = f_6(a, 1, 1) - \frac{27}{2}p^2(a, 1, 1) = \frac{1}{108}(a - 1)^2(2a + 1)^2(19a^2 - 11a + 19) \geq 0.
\]

Thus, the proof is completed. The equality holds for \(a = b = c\), and also for \(-2a = b = c\) (or any cyclic permutation).

\[\square\]

P 1.164. If \(a, b, c\) are real numbers, no two of which are zero, then

\[
\frac{4a^2 + bc}{4b^2 + 7bc + 4c^2} + \frac{4b^2 + ca}{4c^2 + 7ca + 4a^2} + \frac{4c^2 + ab}{4a^2 + 7ab + 4b^2} \geq 1.
\]

(Vasile Cîrtoaje, 2011)
**Solution.** Write the inequality as $f_6(a, b, c) \geq 0$, where
\[
f_6(a, b, c) = \sum (4a^2 + bc)(4a^2 + 7ab + 4b^2)(4a^2 + 7ac + 4c^2) - \prod (4b^2 + 7bc + 4c^2).
\]
Let
\[
p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.
\]
From
\[
f_6(a, b, c) = \sum (4a^2 + bc)(7ab - 4c^2 + 4p^2 - 8q)(7ac - 4b^2 + 4p^2 - 8q)
\]
\[
- \prod (7bc - 4a^2 + 4p^2 - 8q),
\]
it follows that $f_6(a, b, c)$ has the same highest coefficient $A$ as
\[
P_2(a, b, c) - P_3(a, b, c),
\]
where
\[
P_2(a, b, c) = \sum (4a^2 + bc)(7ab - 4c^2),
\]
\[
P_3(a, b, c) = \prod (7bc - 4a^2).
\]
According to Remark 2 from P 1.75 in Volume 1,
\[
A = P_2(1, 1, 1) - P_3(1, 1, 1) = 135 - 27 = 108.
\]
Since $A > 0$, we will apply the highest coefficient cancellation method. Define the homogeneous polynomial
\[
P(a, b, c) = r + Bp^3 + Cpq,
\]
where $B$ and $C$ are real constants. We will show that there are two real numbers $B$ and $C$ such that the following sharper inequality holds
\[
f_6(a, b, c) \geq 108p^2(a, b, c).
\]
Let us denote
\[
g_6(a, b, c) = f_6(a, b, c) - 108p^2(a, b, c).
\]
Clearly, $g_6(a, b, c)$ has the highest coefficient $A_1 = 0$. Then, by P 1.75 in Volume 1, it suffices to prove that $g_6(a, 1, 1) \geq 0$ for all real $a$.

We have
\[
g_6(a, 1, 1) = f_6(a, 1, 1) - 108p^2(a, 1, 1),
\]
where
\[
f_6(a, 1, 1) = 4(4a^2 + 7a + 4)(a - 1)^2(4a^2 + 15a + 16),
\]
\[
P(a, 1, 1) = a + B(a + 2)^3 + C(a + 2)(2a + 1).
Let us denote \( g(a) = f_6(a, 1, 1) \). Since \( g(-2) = 0 \), we can have \( g(a) \geq 0 \) in the vicinity of \( a = -2 \) only if \( g'(-2) = 0 \), which involves \( C = -5/9 \). On the other hand, from \( g(1) = 0 \), we get \( B = 4/27 \). For these values of \( B \) and \( C \), we get

\[
P(a, 1, 1) = \frac{2(a - 1)^2(2a + 1)}{27},
\]

\[
g_6(a, 1, 1) = \frac{4}{27}(a - 1)^2(a + 2)^2(416a^2 + 728a + 431) \geq 0.
\]

The proof is completed. The equality holds for \( a = b = c \), and for \( a = 0 \) and \( b + c = 0 \) (or any cyclic permutation).

\[\square\]

**P 1.165.** If \( a, b, c \) are real numbers, no two of which are equal, then

\[
\frac{1}{(a - b)^2} + \frac{1}{(b - c)^2} + \frac{1}{(c - a)^2} \geq \frac{27}{4(a^2 + b^2 + c^2 - ab - bc - ca)}.
\]

**First Solution.** Write the inequality as follows

\[
\left[ (a - b)^2 + (b - c)^2 + (a - c)^2 \right] \left[ \frac{1}{(a - b)^2} + \frac{1}{(b - c)^2} + \frac{1}{(a - c)^2} \right] \geq \frac{27}{2},
\]

\[
\left[ \frac{(a - b)^2}{(a - c)^2} + \frac{(b - c)^2}{(a - c)^2} + 1 \right] \left[ \frac{(a - c)^2}{(a - b)^2} + \frac{(b - c)^2}{(b - c)^2} + 1 \right] \geq \frac{27}{2},
\]

\[
(x^2 + y^2 + 1) \left( \frac{1}{x^2} + \frac{1}{y^2} + 1 \right) \geq \frac{27}{2},
\]

where

\[
x = \frac{a - b}{a - c}, \quad y = \frac{b - c}{a - c}, \quad x + y = 1.
\]

We have

\[
(x^2 + y^2 + 1) \left( \frac{1}{x^2} + \frac{1}{y^2} + 1 \right) - \frac{27}{2} = \frac{(x + 1)^2(x - 2)^2(2x - 1)^2}{2x^2(1-x)^2} \geq 0.
\]

The proof is completed. The equality holds for \( 2a = b + c \) (or any cyclic permutation).

**Second Solution.** Assume that \( a > b > c \). We have

\[
\frac{1}{(a - b)^2} + \frac{1}{(b - c)^2} \geq \frac{2}{(a - b)(b - c)} \geq \frac{8}{[(a - b) + (b - c)]^2} = \frac{8}{(a - c)^2}.
\]
Therefore, it suffices to show that
\[
\frac{9}{(a-c)^2} \geq \frac{27}{4(a^2 + b^2 + c^2 - ab - bc - ca)},
\]
which is equivalent to
\[
(a - 2b + c)^2 \geq 0.
\]

Third Solution. Write the inequality as
\[
f_6(a, b, c) \geq 0,
\]
where
\[
f_6(a, b, c) = 4(a^2 + b^2 + c^2 - ab - bc - ca) \sum (a-b)^2(a-c)^2 - 27(a-b)^2(b-c)^2(c-a)^2.
\]
Clearly, \(f_6(a, b, c)\) has the same highest coefficient \(A\) as
\[
-27(a-b)^2(b-c)^2(c-a)^2;
\]
that is,
\[
A = -27(-27) = 729.
\]
Since \(A > 0\), we will use the highest coefficient cancellation method. Define the homogeneous polynomial
\[
P(a, b, c) = abc + B(a + b + c)^3 - \left(3B + \frac{1}{9}\right)(a + b + c)(ab + bc + ca),
\]
which satisfies the property \(P(1, 1, 1) = 0\). We will show that there is a real value of \(B\) such that the following sharper inequality holds
\[
f_6(a, b, c) \geq 729P^2(a, b, c).
\]
Let us denote
\[
g_6(a, b, c) = f_6(a, b, c) - 729P^2(a, b, c).
\]
Clearly, \(g_6(a, b, c)\) has the highest coefficient \(A_1 = 0\). Then, by P 1.75 in Volume 1, it suffices to prove that \(g_6(a, 1, 1) \geq 0\) for all real \(a\).
We have
\[
f_6(a, 1, 1) = 4(a - 1)^6
\]
and
\[
P(a, 1, 1) = \frac{1}{9}(a - 1)^2[9B(a + 2) + 2],
\]

hence
\[
g_6(a, 1, 1) = f_6(a, 1, 1) - 729P^2(a, 1, 1) = (27B + 2)(a-1)^4(a+2)[(2-27B)a-54B-8].
\]
Choosing \(B = -2/27\), we get \(g_6(a, 1, 1) = 0\) for all real \(a\).

Remark. The inequality is equivalent to
\[
(a - 2b + c)^2(b - 2c + a)^2(c - 2a + b)^2 \geq 0.
\]
\textbf{P 1.166.} If \(a, b, c\) are real numbers, no two of which are zero, then
\[\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \geq \frac{14}{3(a^2 + b^2 + c^2)}.\]

\textit{(Vasile Cîrtoaje and BJSL, 2014)}

\textbf{Solution.} Write the inequality as \(f_6(a, b, c) \geq 0\), where
\[f_6(a, b, c) = 3(a^2 + b^2 + c^2) \sum (a^2 - ab + b^2)(a^2 - ac + c^2) - 14(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2).\]

Clearly, \(f_6(a, b, c)\) has the same highest coefficient \(A\) as
\[-14(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2),\]

hence as
\[f(a, b, c) = -14(-c^2 - ab)(-a^2 - bc)(-b^2 - ca);\]

that is, according to Remark 2 from P 1.75 in Volume 1,
\[A = f(1, 1, 1) = -14(-2)^2 = 112.\]

Since \(A > 0\), we apply the \textit{highest coefficient cancellation method}. Define the homogeneous polynomial
\[P(a, b, c) = abc + B(a + b + c)^3 + C(a + b + c)(ab + bc + ca)\]

We will show that there are two real numbers \(B\) and \(C\) such that the following sharper inequality holds
\[f_6(a, b, c) \geq 112P^2(a, b, c).\]

Let us denote
\[g_6(a, b, c) = f_6(a, b, c) - 112P^2(a, b, c).\]

Clearly, \(g_6(a, b, c)\) has the highest coefficient \(A_1 = 0\). By P 1.75 in Volume 1, it suffices to prove that \(g_6(a, 1, 1) \geq 0\) for all real \(a\).

We have
\[g_6(a, 1, 1) = f_6(a, 1, 1) - 112P^2(a, 1, 1),\]

where
\[f_6(a, 1, 1) = (a^2 - a + 1)(3a^4 - 3a^3 + a^2 + 8a + 4),\]
\[P(a, 1, 1) = 1 + B(a + 2)^3 + C(a + 2)(2a + 1).\]

Let us denote \(g(a) = g_6(a, 1, 1)\). Since \(g(-2) = 0\), we can have \(g(a) \geq 0\) in the vicinity of \(a = -2\) only if \(g'(-2) = 0\), which involves \(C = -4/7\). In addition, setting \(B = 9/56\), we get
\[P(a, 1, 1) = \frac{1}{56}(9a^3 - 10a^2 + 4a + 8),\]
\[ g_6(a, 1, 1) = \frac{3}{28} (a^6 + 4a^5 + 8a^4 + 16a^3 + 20a^2 + 16a + 16) \]
\[ = \frac{3(a + 2)^2(a^2 + 2)^2}{28} \geq 0. \]

The proof is completed. The equality holds for \( a = 0 \) and \( b + c = 0 \) (or any cyclic permutation).

\[ \square \]

**P 1.167.** Let \( a, b, c \) be real numbers such that \( ab + bc + ca \geq 0 \) and no two of which are zero. Prove that

\( a \)
\[
\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2};
\]

\( b \) if \( ab \leq 0 \), then
\[
\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq 2.
\]

*(Vasile Cîrtoaje, 2014)*

**Solution.** Let us show first that \( b + c \neq 0 \), \( c + a \neq 0 \) and \( a + b \neq 0 \). Indeed, if \( b + c = 0 \), then \( ab + bc + ca \geq 0 \) yields \( b = c = 0 \), which is not possible.

(a) Write the inequality as follows

\[
\sum \left( \frac{a}{b + c} + 1 \right) \geq \frac{9}{2},
\]

\[
\left[ \sum (b + c) \right] \left( \sum \frac{1}{b + c} \right) \geq 9,
\]

\[
\sum \left( \frac{a + b}{a + c} + \frac{a + c}{a + b} - 2 \right) \geq 0,
\]

\[
\sum \frac{(b - c)^2}{(a + b)(a + c)} \geq 0,
\]

\[
\sum \frac{(b - c)^2}{a^2 + (ab + bc + ca)} \geq 0.
\]

Clearly, the last inequality is true. The equality holds for \( a = b = c \neq 0 \).

(b) From \( ab + bc + ca \geq 0 \), it follows that if one of \( a, b, c \) is zero, then the others are the same sign. In this case, the desired inequality is trivial. So, due to symmetry and homogeneity, it suffices to consider that \( a < 0 < b \leq c \).
**First Solution.** We will show that

\[ F(a, b, c) > F(0, b, c) \geq 2, \]

where

\[ F(a, b, c) = \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}. \]

We have

\[ F(0, b, c) = \frac{b}{c} + \frac{c}{b} \geq 2 \]

and

\[ F(a, b, c) - F(0, b, c) = a \left[ \frac{1}{b + c} - \frac{b}{c(c + a)} - \frac{c}{b(a + b)} \right]. \]

Since \( a < 0 \), we need to show that

\[ \frac{b}{c(c + a)} + \frac{c}{b(a + b)} > \frac{1}{b + c}. \]

From \( ab + bc + ca \geq 0 \), we get

\[ c + a \geq \frac{-ca}{b} > 0, \quad a + b \geq \frac{-ab}{c} > 0, \]

hence

\[ \frac{b}{c(c + a)} > \frac{b}{c^2}, \quad \frac{c}{b(a + b)} > \frac{c}{b^2}. \]

Therefore, it suffices to prove that

\[ \frac{b}{c^2} + \frac{c}{b^2} \geq \frac{1}{b + c}. \]

Indeed, by virtue of the AM-GM inequality, we have

\[ \frac{b}{c^2} + \frac{c}{b^2} - \frac{1}{b + c} \geq \frac{2}{\sqrt{bc}} - \frac{1}{2\sqrt{bc}} > 0. \]

This completes the proof. The equality holds for \( a = 0 \) and \( b = c \), or \( b = 0 \) and \( a = c \).

**Second Solution.** Since \( b + c > 0 \) and

\[ (b + c)(a + b) = b^2 + (ab + bc + ca) > 0, \quad (b + c)(c + a) = c^2 + (ab + bc + ca) > 0, \]

we get \( a + b > 0 \) and \( c + a > 0 \). By virtue of the Cauchy-Schwarz inequality and AM-GM inequality, we have
\[
\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{a}{b + c} + \frac{(b + c)^2}{b(c + a) + c(a + b)}
\]

\[
> \frac{a}{2a + b + c} + \frac{(b + c)^2}{2(b + c)} + a(b + c)
\]

\[
> \frac{4a}{2a + b + c} + \frac{2(b + c)}{2a + b + c} = 2.
\]

\[ \square \]

P 1.168. If \( a, b, c \) are nonnegative real numbers, then

\[
\frac{a}{7a + b + c} + \frac{b}{7b + c + a} + \frac{c}{7c + a + b} \geq \frac{ab + bc + ca}{(a + b + c)^2}.
\]

(\textit{Vasile Cîrtoaje, 2014})

First Solution. Write the inequality as follows:

\[
\sum \left[ \frac{2a}{7a + b + c} - \frac{a(b + c)}{(a + b + c)^2} \right] \geq 0,
\]

\[
\sum \frac{a[(a - b) + (a - c)](a - b - c)}{7a + b + c} \geq 0,
\]

\[
\sum \frac{a(a - b)(a - b - c)}{7a + b + c} + \sum \frac{a(a - c)(a - b - c)}{7a + b + c} \geq 0,
\]

\[
\sum \frac{a(a - b)(a - b - c)}{7a + b + c} + \sum \frac{b(b - a)(b - c - a)}{7b + c + a} \geq 0,
\]

\[
\sum (a - b) \left[ \frac{a(a - b - c)}{7a + b + c} - \frac{b(b - c - a)}{7b + c + a} \right] \geq 0,
\]

\[
\sum (a - b)^2(a^2 + b^2 - c^2 + 14ab)(a + b + 7c) \geq 0.
\]

Since

\[
a^2 + b^2 - c^2 + 14ab \geq (a + b)^2 - c^2 = (a + b + c)(a + b - c),
\]

it suffices to show that

\[
\sum (a - b)^2(a + b - c)(a + b + 7c) \geq 0.
\]

Assume that \( a \geq b \geq c \). It suffices to show that

\[
(a - c)^2(a - b + c)(a + 7b + c) + (b - c)^2(-a + b + c)(7a + b + c) \geq 0.
\]
Symmetric Rational Inequalities

For the nontrivial case \( b > 0 \), we have

\[
(a-c)^2 \geq \frac{a^2}{b^2}(b-c)^2 \geq \frac{a}{b}(b-c)^2.
\]

Thus, it is enough to prove that

\[
a(a-b+c)(a+7b+c)+b(-a+b+c)(7a+b+c) \geq 0.
\]

Since

\[
a(a+7b+c) \geq b(7a+b+c),
\]

we have

\[
a(a-b+c)(a+7b+c)+b(-a+b+c)(7a+b+c) \geq
\]

\[
\geq b(a-b+c)(7a+b+c)+b(-a+b+c)(7a+b+c)
\]

\[
= 2bc(7a+b+c) \geq 0.
\]

This completes the proof. The equality holds for \( a = b = c \), and also for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

Second Solution. Assume that \( a \leq b \leq c \), \( a+b+c = 3 \) and use the substitution

\[
x = \frac{2a+1}{3}, \quad y = \frac{2b+1}{3}, \quad z = \frac{2c+1}{3},
\]

where \( 1/3 \leq x \leq y \leq z \), \( x+y+z = 3 \). We have \( b+c \geq 2 \), \( y+z \geq 2 \), \( x \leq 1 \). The inequality becomes

\[
\frac{a}{2a+1} + \frac{b}{2b+1} + \frac{c}{2c+1} \geq \frac{9-a^2-b^2-c^2}{6},
\]

\[
9(x^2+y^2+z^2) \geq 4\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + 17.
\]

Assume that \( x \leq y \leq z \) and show that

\[
E(x, y, z) \geq E(x, t, t) \geq 0,
\]

where

\[
t = (y+z)/2 = (3-x)/2
\]

and

\[
E(x, y, z) = 9(x^2+y^2+z^2) - 4\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 17.
\]
We have

\[ E(x, y, z) - E(x, t, t) = 9(y^2 + z^2 - 2t^2) - 4 \left( \frac{1}{y} + \frac{1}{z} - \frac{2}{t} \right) \]

\[ = \frac{(y - z)^2[9yz(y + z) - 8]}{2yz(y + z)} \geq 0, \]

since

\[ 9yz = (2b + 1)(2c + 1) \geq 2(b + c) + 1 \geq 5, \quad y + z \geq 2. \]

Also,

\[ E(x, t, t) = 9x^2 + 2t^2 - 15 - \frac{4}{x} - \frac{8}{t} = \frac{(x - 1)^2(3x - 1)(8 - 3x)}{2x(3 - x)} \geq 0. \]

**Third Solution.** Write the inequality as \( f_5(a, b, c) \geq 0 \), where \( f_5(a, b, c) \) is a symmetric homogeneous inequality of degree five. According to P 2.68-(a) in Volume 1, it suffices to prove the inequality for \( a = 0 \) and for \( b = c = 1 \). For \( a = 0 \), the inequality is equivalent to

\[ (b - c)^2(b^2 + c^2 + 11bc) \geq 0, \]

while, for \( b = c = 1 \), the inequality is equivalent to

\[ a(a - 1)^2(a + 14) \geq 0. \]

\[ \square \]

**P 1.169.** If \( a, b, c \) are the lengths of the sides of a triangle, then

\[ \frac{a^2}{4a^2 + 5bc} + \frac{b^2}{4b^2 + 5ca} + \frac{c^2}{4c^2 + 5ab} \geq \frac{1}{3}. \]

(Vasile Cîrtoaje, 2009)

**Solution.** Write the inequality as \( f_6(a, b, c) \geq 0 \), where

\[ f_6(a, b, c) = 3 \sum a^2(4b^2 + 5ca)(4c^2 + 5ab) - \prod (4a^2 + 5bc) \]

\[ = -45a^2b^2c^2 - 25abc \sum a^3 + 40 \sum a^3b^3. \]

Since \( f_6(a, b, c) \) has the highest coefficient

\[ A = -45 - 75 + 120 = 0, \]
according to 2.67-(b), it suffices to prove the original inequality \( b = c = 1 \) and \( 0 \leq a \leq 2 \), and for \( a = b + c \).

**Case 1:** \( b = c = 1 \), \( 0 \leq a \leq 2 \). The original inequality reduces to

\[
(2 - a)(a - 1)^2 \geq 0,
\]

which is true.

**Case 2:** \( a = b + c \). Using the Cauchy-Schwarz inequality

\[
\frac{b^2}{4b^2 + 5ca} + \frac{c^2}{4c^2 + 5ab} \geq \frac{(b + c)^2}{4(b^2 + c^2) + 5a(b + c)},
\]

it suffices to show that

\[
\frac{a^2}{4a^2 + 5bc} + \frac{(b + c)^2}{4(b^2 + c^2) + 5a(b + c)} \geq \frac{1}{3},
\]

which reduces to the obvious inequality

\[
(b - c)^2(3b^2 + 3c^2 - 4bc) \geq 0.
\]

The equality holds for an equilateral triangle, and for a degenerate triangle with \( a/2 = b = c \) (or any cyclic permutation).

\( \square \)

**P 1.170.** If \( a, b, c \) are the lengths of the sides of a triangle, then

\[
\frac{1}{7a^2 + b^2 + c^2} + \frac{1}{7b^2 + c^2 + a^2} + \frac{1}{7c^2 + a^2 + b^2} \geq \frac{3}{(a + b + c)^2},
\]

(Vo Quoc Ba Can, 2010)

**Solution.** Let \( p = a + b + c \) and \( q = ab + bc + ca \). Write the inequality as \( f_6(a, b, c) \geq 0 \), where

\[
f_6(a, b, c) = p^2 \sum(7b^2 + c^2 + a^2)(7c^2 + a^2 + b^2) - 3 \prod(7a^2 + b^2 + c^2)
\]

\[
= p^2 \sum(6b^2 + p^2 - 2q)(6c^2 + p^2 - 2q) - 3 \prod(6a^2 + p^2 - 2q).
\]

Since \( f_6(a, b, c) \) has the highest coefficient

\[
A = -3(6^3) < 0,
\]

according to 2.67-(b), it suffices to prove the original inequality \( b = c = 1 \) and \( 0 \leq a \leq 2 \), and for \( a = b + c \).
Case 1: \( b = c = 1, 0 \leq a \leq 2 \). The original inequality reduces to
\[
a(8-a)(a-1)^2 \geq 0,
\]
which is true.

Case 2: \( a = b + c \). Write the inequality as
\[
\frac{1}{4(b^2 + c^2) + 7bc} + \frac{1}{4b^2 + c^2 + bc} + \frac{1}{4c^2 + b^2 + bc} \geq \frac{3}{2(b + c)^2},
\]
\[
\frac{1}{4x + 7} + \frac{5x + 2}{4x^2 + 5x + 10} \geq \frac{3}{2(x + 2)},
\]
where \( x = b/c + c/b, x \geq 2 \). This inequality is equivalent to the obvious inequality
\[
16x^2 + 5x - 38 \geq 0.
\]
The equality holds for an equilateral triangle, and for a degenerate triangle with \( a = 0 \) and \( b = c \) (or any cyclic permutation).
\( \square \)

P 1.171. Let \( a, b, c \) be the lengths of the sides of a triangle. If \( k > -2 \), then
\[
\sum a(b + c) + (k + 1)bc \leq \frac{3(k + 3)}{k + 2}.
\]
(Vasile Cîrtoaje, 2009)

Solution. Let \( p = a + b + c \) and \( q = ab + bc + ca \). Write the inequality as \( f_6(a, b, c) \geq 0 \), where
\[
f_6(a, b, c) = 3(k + 3) \prod (b^2 + kbc + c^2)
\]
\[-(k + 2) \sum [a(b + c) + (k + 1)bc](c^2 + kca + a^2)(a^2 + kab + b^2)
\]
\[= 3(k + 3) \prod (p^2 - 2q + kbc - a^2)
\]
\[-(k + 2) \sum (q + kbc)(p^2 - 2q + kca - b^2)(p^2 - 2q + kab - c^2).
\]
Since \( f_6(a, b, c) \) has the same highest coefficient \( A \) as \( f(a, b, c) \), where
\[
f(a, b, c) = 3(k + 3) \prod (kbc - a^2) - k(k + 2) \sum bc(kca - b^2)(kab - c^2)
\]
\[
= 3(k + 3)[(k^3 - 1)a^2b^2c^2 - k^2abc \sum a^3 + k \sum a^3b^2]
\]
\[-k(k + 2)(3k^2a^2b^2c^2 - 2kabc \sum a^3 + \sum a^3b^3),
\]
we get
\[ A = 3(k + 3)(k^3 - 1 - 3k^2 + 3k) - k(k + 2)(3k^2 - 6k + 3) = -9(k - 1)^2 \leq 0. \]

According to 2.67-(b), it suffices to prove the original inequality \( b = c = 1 \) and \( 0 \leq a \leq 2 \), and for \( a = b + c \).

Case 1: \( b = c = 1, 0 \leq a \leq 2 \). The original inequality reduces to
\[ (2-a)(a-1)^2 \geq 0, \]
which is true.

Case 2: \( a = b + c \). Write the inequality as follows
\[
\sum \left[ a(b+c) + (k+1)bc \right] \leq \frac{3}{k+2},
\]
\[
\sum \left( \frac{ab + bc + ca - b^2 - c^2}{b^2 + kbc + c^2} \right) \leq \frac{3}{k+2},
\]
\[
\frac{3bc}{b^2 + kbc + c^2} + \frac{bc - c^2}{b^2 + (k+2)(bc + c^2)} + \frac{bc - b^2}{c^2 + (k+2)(bc + b^2)} \leq \frac{3}{k+2}.
\]

Since
\[
\frac{3bc}{b^2 + kbc + c^2} \leq \frac{3}{k+2},
\]
it suffices to prove that
\[
\frac{bc - c^2}{b^2 + (k+2)(bc + c^2)} + \frac{bc - b^2}{c^2 + (k+2)(bc + b^2)} \leq 0.
\]

This reduces to the obvious inequality
\[
(b-c)^2(b^2 + bc + c^2) \geq 0.
\]

The equality holds for an equilateral triangle, and for a degenerate triangle with \( a/2 = b = c \) (or any cyclic permutation).

\[ \square \]

**P 1.172.** Let \( a, b, c \) be the lengths of the sides of a triangle. If \( k > -2 \), then
\[
\sum \frac{2a^2 + (4k + 9)bc}{b^2 + kbc + c^2} \leq \frac{3(4k + 11)}{k + 2}.
\]

(\textit{Vasile Cirtoaje, 2009})
Solution. Let $p = a + b + c$ and $q = ab + bc + ca$. Write the inequality as $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = 3(4k + 11) \prod (b^2 + kbc + c^2)$$

$$-(k + 2) \sum [2a^2 + (4k + 9)bc](c^2 + kca + a^2)(a^2 + kab + b^2)$$

$$= 3(4k + 11) \prod (p^2 - 2q + kbc - a^2)$$

$$-(k + 2) \sum [2a^2 + (4k + 9)bc](p^2 - 2q + kca - b^2)(p^2 - 2q + kab - c^2).$$

Since $f_6(a, b, c)$ has the same highest coefficient $A$ as $f(a, b, c)$, where

$$f(a, b, c) = 3(4k + 11) \prod (kbc - a^2)$$

$$-(k + 2) \sum [2a^2 + (4k + 9)bc](kca - b^2)(kab - c^2)$$

$$= 3(4k + 11)[(k^3 - 1)a^2b^2c^2 - k^2abc \sum a^3 + k \sum a^3b^3]$$

$$-(k + 2)[3(4k^3 + 9k^2 + 2)a^2b^2c^2 - 6k(k + 3)abc \sum a^3 + 9 \sum a^3b^3],$$

we get

$$A = 3(4k + 11)(k^3 - 1 - 3k^2 + 3k) - (k + 2)[3(4k^3 + 9k^2 + 2) - 18k(k + 3) + 27]$$

$$= -9(4k + 11)(k - 1)^2 \leq 0.$$ 

According to 2.67-(b), it suffices to prove the original inequality $b = c = 1$ and $0 \leq a \leq 2$, and for $a = b + c$.

Case 1: $b = c = 1$, $0 \leq a \leq 2$. The original inequality reduces to

$$(2 - a)(a - 1)^2 \geq 0,$$

which is true.

Case 2: $a = b + c$. Write the inequality as follows

$$\sum \left[ \frac{2a^2 + (4k + 9)bc}{b^2 + kbc + c^2} - 2 \right] \leq \frac{3(2k + 7)}{k + 2},$$

$$\sum \frac{2(a^2 - b^2 - c^2) + (2k + 9)bc}{b^2 + kbc + c^2} \leq \frac{3(2k + 7)}{k + 2},$$

$$\frac{(2k + 13)bc}{b^2 + kbc + c^2} + (2k + 5)(b + c)\left[ \frac{c}{b^2 + (k + 2)(bc + c^2)} + \frac{b}{c^2 + (k + 2)(bc + b^2)} \right] \leq \frac{3(2k + 7)}{k + 2}.$$
Using the substitution \( x = b/c + c/b, \ x \geq 2 \), the inequality can be written as
\[
\frac{2k + 13}{x + k} + \frac{(2k + 5)(x + 2)(x + 2k + 3)}{(k + 2)x^2 + (k + 2)(k + 3)x + 2k^2 + 6k + 5} \leq \frac{3(2k + 7)}{k + 2},
\]
where
\[
(x - 2)(4(k + 2)Ax^2 + 2(k + 2)Bx + C) \geq 0,
\]
where
\[
A = k + 4, \quad B = 2k^2 + 13k + 22, \quad C = 8k^3 + 51k^2 + 98k + 65.
\]
The inequality is true since \( A > 0, \ B = 2(k + 2)^2 + 5(k + 2) + 4 > 0, \)
\[
C = 8(k + 2)^3 + 2k^2 + (k + 1)^2 > 0.
\]
The equality holds for an equilateral triangle, and for a degenerate triangle with \( a/2 = b = c \) (or any cyclic permutation).

\[\square\]

**P 1.173.** If \( a \geq b \geq c \geq d \) such that \( abcd = 1 \), then
\[
\frac{1}{1 + a} + \frac{1}{1 + b} + \frac{1}{1 + c} \geq \frac{3}{1 + \sqrt[3]{abc}}.
\]

*(Vasile Cîrtoaje, 2008)*

**Solution.** We can get this inequality by summing the inequalities below
\[
\frac{1}{1 + a} + \frac{1}{1 + b} \geq \frac{2}{1 + \sqrt{ab}},
\]
\[
\frac{1}{1 + c} + \frac{2}{1 + \sqrt{ab}} \geq \frac{3}{1 + \sqrt{abc}}.
\]
The first inequality is true, since
\[
\frac{1}{1 + a} + \frac{1}{1 + b} - \frac{2}{1 + \sqrt{ab}} = \left(\frac{1}{1 + a} - \frac{1}{1 + \sqrt{ab}}\right) + \left(\frac{1}{1 + b} - \frac{1}{1 + \sqrt{ab}}\right) = \frac{(\sqrt{a} - \sqrt{b})^2(\sqrt{ab} - 1)}{(1 + a)(1 + b)(1 + \sqrt{ab})}
\]
and \( ab \geq \sqrt{abcd} = 1 \). To prove the second inequality, we denote \( x = \sqrt{ab} \) and \( y = \sqrt[3]{abc} \) \((x \geq y \geq 1)\), which yield \( c = y^3/x^2 \). From \( abc^2 \geq abcd = 1 \), we get \( abc \geq \sqrt{ab} \), that is, \( y^3 \geq x \). Since
\[
\frac{1}{1 + c} + \frac{2}{1 + \sqrt{ab}} - \frac{3}{1 + \sqrt[3]{abc}} = \frac{x^2}{x^2 + y^3} + \frac{2}{1 + x} - \frac{3}{1 + y}
\]
\[
= \left( \frac{x^2}{x^2 + y^3} - \frac{1}{1+y} \right) + 2 \left( \frac{1}{1+x} - \frac{1}{1+y} \right)
\]
\[
= \frac{(x-y)^2[(y-2)x + 2y^2 - y]}{(1+x)(1+y)(x^2 + y^3)},
\]

we still have to show that \((y-2)x + 2y^2 - y \geq 0\). This is clearly true for \(y \geq 2\), while for \(1 \leq y < 2\), we have
\[
(y-2)x + 2y^2 - y \geq (y-2)y^2 + 2y^2 - y = y(y-1)(y^2 - y + 1) \geq 0.
\]
The equality holds for \(a = b = c\).

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\[\text{P 1.174. Let } a, b, c, d \text{ be positive real numbers such that } abcd = 1. \text{ Prove that}
\]
\[
\sum \frac{1}{1 + ab + bc + ca} \leq 1.
\]

\[\text{Solution.}\]

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}} = \sqrt{d}(\sqrt{a} + \sqrt{b} + \sqrt{c}),
\]

we get
\[
ab + bc + ca \geq \frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{\sqrt{d}}.
\]

Therefore,
\[
\sum \frac{1}{1 + ab + bc + ca} \leq \sum \frac{\sqrt{d}}{\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}} = 1,
\]

which is just the required inequality. The equality occurs for \(a = b = c = d = 1\).  

\[\text{P 1.175. Let } a, b, c, d \text{ be positive real numbers such that } abcd = 1. \text{ Prove that}
\]
\[
\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} + \frac{1}{(1 + c)^2} + \frac{1}{(1 + d)^2} \geq 1.
\]

\[(\text{Vasile Cirtoaje, 1995)}\]
First Solution. The inequality follows by summing the following inequalities (see the proof of P 1.45):

\[
\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab},
\]

\[
\frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq \frac{1}{1+cd} = \frac{ab}{1+ab}.
\]

The equality occurs for \(a = b = c = d = 1\).

Second Solution. Using the substitutions \(a = 1/x^4, b = 1/y^4, c = 1/z^4, d = 1/t^4\), where \(x, y, z, t\) are positive real numbers such that \(xyzt = 1\), the inequality becomes as follows

\[
\frac{x^6}{(x^3 + 1/x)^2} + \frac{y^6}{(y^3 + 1/y)^2} + \frac{z^6}{(z^3 + 1/z)^2} + \frac{t^6}{(t^3 + 1/t)^2} \geq 1.
\]

By the Cauchy-Schwarz inequality, we get

\[
\sum \frac{x^6}{(x^3 + 1/x)^2} \geq \frac{(\sum x^3)^2}{\sum (x^3 + 1/x)^2} = \frac{(\sum x^3)^2}{\sum x^6 + 2\sum x^2 + \sum x^2y^2z^2}.
\]

Thus, it suffices to show that

\[
2(x^3y^3 + x^3z^3 + x^3t^3 + y^3z^3 + y^3t^3 + z^3t^3) \geq 2xyzt \sum x^2 + \sum x^2y^2z^2.
\]

This is true if

\[
2(x^3y^3 + x^3z^3 + x^3t^3 + y^3z^3 + y^3t^3 + z^3t^3) \geq 3xyzt \sum x^2
\]

and

\[
2(x^3y^3 + x^3z^3 + x^3t^3 + y^3z^3 + y^3t^3 + z^3t^3) \geq 3 \sum x^2y^2z^2.
\]

Write these inequalities as

\[
\sum x^3(y^3 + z^3 + t^3 - 3yzt) \geq 0
\]

and

\[
\sum (x^3y^3 + y^3z^3 + z^3x^3 - 3x^2y^2z^2) \geq 0,
\]

respectively. By the AM-GM inequality, we have

\[
y^3 + z^3 + t^3 \geq 3yzt\quad \text{and}\quad x^3y^3 + y^3z^3 + z^3x^3 \geq 3x^2y^2z^2.
\]

Thus the conclusion follows.

Third Solution. Using the substitutions \(a = yz/x^2, b = zt/y^2, c = tx/z^2, d = xy/t^2\), where \(x, y, z, t\) are positive real numbers, the inequality becomes

\[
\frac{x^4}{(x^2 + yz)^2} + \frac{y^4}{(y^2 + zt)^2} + \frac{z^4}{(z^2 + tx)^2} + \frac{t^4}{(t^2 + xy)^2} \geq 1.
\]
Using the Cauchy-Schwarz inequality two times, we deduce
\[ \frac{x^4}{(x^2 + yz)^2} + \frac{z^4}{(z^2 + tx)^2} \geq \frac{x^4}{(x^2 + y^2)(x^2 + z^2)} + \frac{z^4}{(z^2 + t^2)(z^2 + x^2)} \]
and hence
\[ \frac{x^4}{(x^2 + y^2)^2} + \frac{z^4}{(z^2 + t^2)^2} \geq \frac{x^2 + z^2}{x^2 + y^2 + z^2 + t^2}. \]

Adding this to the similar inequality
\[ \frac{y^4}{(y^2 + zt)^2} + \frac{t^4}{(t^2 + xy)^2} \geq \frac{y^2 + t^2}{x^2 + y^2 + z^2 + t^2}, \]
we get the required inequality.

**Fourth Solution.** Using the substitutions \( a = x/y, \ b = y/z, \ c = z/t, \ d = t/x, \) where \( x, y, z, t \) are positive real numbers, the inequality can be written as
\[ \frac{y^2}{(x + y)^2} + \frac{z^2}{(y + z)^2} + \frac{t^2}{(z + t)^2} + \frac{x^2}{(t + x)^2} \geq 1. \]

By the Cauchy-Schwarz inequality and the AM-GM inequality, we get
\[ \sum \frac{y^2}{(x + y)^2} \geq \frac{[\sum y(y + z)]^2}{\sum (x + y)^2(y + z)^2} = \frac{[(x + y)^2 + (y + z)^2 + (z + t)^2 + (t + x)^2]^2}{4[(x + y)^2 + (y + z)^2 + (z + t)^2 + (t + x)^2]} \geq 1. \]

**Remark.** The following generalization holds true (Vasile Cîrtoaje, 2005):
- Let \( a_1, a_2, \ldots, a_n \) be positive real numbers such that \( a_1 a_2 \cdots a_n = 1. \) If \( k \geq \sqrt{n} - 1, \) then
\[ \frac{1}{(1 + ka_1)^2} + \frac{1}{(1 + ka_2)^2} + \cdots + \frac{1}{(1 + ka_n)^2} \geq \frac{n}{(1 + k)^2}. \]

\[ \square \]

**P 1.176.** Let \( a, b, c, d \neq \frac{1}{3} \) be positive real numbers such that \( abcd = 1. \) Prove that
\[ \frac{1}{(3a - 1)^2} + \frac{1}{(3b - 1)^2} + \frac{1}{(3c - 1)^2} + \frac{1}{(3d - 1)^2} \geq 1. \]

(Vasile Cîrtoaje, 2006)
**First Solution.** It suffices to show that

\[
\frac{1}{(3a - 1)^2} \geq \frac{a^{-3}}{a^{-3} + b^{-3} + c^{-3} + d^{-3}}.
\]

This inequality is equivalent to

\[
6a^{-2} + b^{-3} + c^{-3} + d^{-3} \geq 9a^{-1},
\]

which follows from the AM-GM inequality, as follows

\[
6a^{-2} + b^{-3} + c^{-3} + d^{-3} \geq 9\sqrt[9]{a^{-12}b^{-3}c^{-3}d^{-3}} = 9a^{-1}.
\]

The equality occurs for \(a = b = c = d = 1\).

**Second Solution.** Let \(a \leq b \leq c \leq d\). If \(a < 1/3\), then

\[
\frac{1}{(3a - 1)^2} > 1,
\]

and the desired inequality is clearly true. Otherwise, if \(1/3 < a \leq b \leq c \leq d\), we have

\[
4a^3 - (3a - 1)^2 = (a - 1)^2(4a - 1) \geq 0.
\]

Therefore, using this result and the AM-GM inequality, we get

\[
\sum \frac{1}{(3a - 1)^2} \geq \frac{1}{4} \sum \frac{1}{a^3} \geq \sqrt[4]{\frac{1}{a^3b^3c^3d^3}} = 1.
\]

**Third Solution.** We have

\[
\frac{1}{(3a - 1)^2} - \frac{1}{(a^3 + 1)^2} = \frac{a(a - 1)^2(a + 2)(a^2 + 3)}{(3a - 1)^2(a^3 + 1)^2} \geq 0.
\]

Therefore,

\[
\sum \frac{1}{(3a - 1)^2} \geq \sum \frac{1}{(a^3 + 1)^2},
\]

and it suffices to prove that

\[
\sum \frac{1}{(a^3 + 1)^2} \geq 1.
\]

This inequality is an immediate consequence of the inequality in P 1.175.

\[
\Box
\]
P 1.177. Let \(a, b, c, d\) be positive real numbers such that \(abcd = 1\). Prove that

\[
\frac{1}{1 + a + a^2 + a^3} + \frac{1}{1 + b + b^2 + b^3} + \frac{1}{1 + c + c^2 + c^3} + \frac{1}{1 + d + d^2 + d^3} \geq 1.
\]

(Vasile Cîrtoaje, 1999)

**First Solution.** We get the desired inequality by summing the inequalities

\[
\frac{1}{1 + a + a^2 + a^3} \geq \frac{1}{1 + (ab)^{3/2}},
\]

\[
\frac{1}{1 + c + c^2 + c^3} \geq \frac{1}{1 + (cd)^{3/2}}.
\]

Thus, it suffices to show that

\[
\frac{1}{1 + x^2 + x^4 + x^6} + \frac{1}{1 + y^2 + y^4 + y^6} \geq \frac{1}{1 + x^3y^3},
\]

where \(x\) and \(y\) are positive real numbers. Putting \(p = xy\) and \(s = x^2 + xy + y^2\), this inequality becomes

\[
p^3(x^6 + y^6) + p^2(p - 1)(x^4 + y^4) - p^2(p^2 - p + 1)(x^2 + y^2) - p^4 - 2p^3 - p^2 + 1 \geq 0,
\]

\[
p^3(x^3 - y^3)^2 + p^2(p - 1)(x^2 - y^2)^2 - p^2(p^2 - p + 1)(x - y)^2 - p^6 - p^4 - p^2 + 1 \geq 0.
\]

\[
p^3s^2(x - y)^2 + p^2(p - 1)(s + p)^2(x - y)^2 - p^2(p^2 - p + 1)(x - y)^2 - p^6 - p^4 - p^2 + 1,
\]

\[
p^2(s + 1)(ps - 1)(x - y)^2 + (p^2 - 1)(p^4 - 1) \geq 0.
\]

If \(ps - 1 \geq 0\), then this inequality is clearly true. Consider further that \(ps < 1\). From \(ps < 1\) and \(s \geq 3p\), we get \(p^2 < 1/3\). Write the desired inequality in the form

\[
(1 - p^2)(1 - p^4) \geq p^2(1 + s)(1 - ps)(x - y)^2.
\]

Since

\[
p(x - y)^2 = p(s - 3p) < 1 - 3p^2 < 1 - p^2,
\]

it suffices to show that

\[
1 - p^4 \geq p(1 + s)(1 - ps).
\]

Indeed,

\[
4p(1 + s)(1 - ps) \leq [p(1 + s) + (1 - ps)]^2 = (1 + p)^2 < 2(1 + p^2) < 4(1 - p^4).
\]

The equality occurs for \(a = b = c = d = 1\).

**Second Solution.** Assume that \(a \geq b \geq c \geq d\), and write the inequality as

\[
\sum_{k=1}^{4} \frac{1}{1 + a} \geq 1.
\]
Since
\[
\frac{1}{1+a} \leq \frac{1}{1+b} \leq \frac{1}{1+c}, \quad \frac{1}{1+a^2} \leq \frac{1}{1+b^2} \leq \frac{1}{1+c^2},
\]
by Chebyshev's inequality, it suffices to prove that
\[
\frac{1}{3} \left( \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \right) \left( \frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} \right) + \frac{1}{1+(d)(1+d^2)} \geq 1.
\]
In addition, from the inequality in P 1.173, we have
\[
\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{3\sqrt[3]{abc}}{1 + \sqrt[3]{abc}} = \frac{3\sqrt[3]{d}}{\sqrt[3]{d} + 1}
\]
and
\[
\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} \geq \frac{3\sqrt[3]{a^2b^2c^2}}{1 + \sqrt[3]{a^2b^2c^2}} = \frac{3\sqrt[3]{d^2}}{\sqrt[3]{d^2} + 1}.
\]
Thus, it suffices to prove that
\[
\frac{3d}{(1 + \sqrt[3]{d})(1 + \sqrt[3]{d^2})} + \frac{1}{(1+d)(1+d^2)} \geq 1.
\]
Putting \( x = \sqrt[3]{d} \), this inequality becomes as follows
\[
\frac{3x^3}{(1 + x)(1 + x^2)} + \frac{1}{(1 + x^3)(1 + x^6)} \geq 1,
\]
\[
3x^3(1-x+x^2)(1-x^2+x^4) + 1 \geq (1+x^3)(1+x^6),
\]
\[
x^3(2-3x+2x^3-3x^5+2x^6) \geq 0,
\]
\[
x^3(1-x)^2(2+x+x^3+2x^4) \geq 0.
\]

**Remark.** The following generalization holds true (Vasile Cirtoaje, 2004):

- If \( a_1, a_2, \ldots, a_n \) are positive real numbers such that \( a_1a_2\cdots a_n = 1 \), then
\[
\frac{1}{1+a_1 + \cdots + a_1^{n-1}} + \frac{1}{1+a_2 + \cdots + a_2^{n-1}} + \cdots + \frac{1}{1+a_n + \cdots + a_n^{n-1}} \geq 1.
\]

\[
\P 1.178. \text{Let } a, b, c, d \text{ be positive real numbers such that } abcd = 1. \text{ Prove that}
\[
\frac{1}{1+a + 2a^2} + \frac{1}{1+b + 2b^2} + \frac{1}{1+c + 2c^2} + \frac{1}{1+d + 2d^2} \geq 1.
\]
\( \text{(Vasile Cirtoaje, 2006)} \)
Solution. We will show that
\[
\frac{1}{1 + a + 2a^2} \geq \frac{1}{1 + a^k + a^{2k} + a^{3k}},
\]
where \( k = 5/6 \). Then, it suffices to show that
\[
\sum \frac{1}{1 + a^k + a^{2k} + a^{3k}} \geq 1,
\]
which immediately follows from the inequality in P 1.177. Setting \( a = x^6 \), \( x > 0 \), the claimed inequality can be written as
\[
\frac{1}{2x^{12} + x^6 + 1} \geq \frac{1}{1 + x^5 + x^{10} + x^{15}},
\]
which is equivalent to
\[
x^{10} + x^5 + 1 \geq 2x^7 + x.
\]
We can prove it by summing the AM-GM inequalities
\[
x^5 + 4 \geq 5x
\]
and
\[
5x^{10} + 4x^5 + 1 \geq 10x^7.
\]
This completes the proof. The equality occurs for \( a = b = c = d = 1 \).

Remark. The inequalities in P 1.175, P 1.177 and P 1.178 are particular cases of the following more general inequality (Vasile Cîrtoaje, 2009):

- Let \( a_1, a_2, \ldots, a_n (n \geq 4) \) be positive real numbers such that \( a_1a_2 \cdots a_n = 1 \). If \( p, q, r \) are nonnegative real numbers satisfying \( p + q + r = n - 1 \), then
\[
\sum_{i=1}^{i=n} \frac{1}{1 + pa_i + qa_i^2 + ra_i^3} \geq 1.
\]

\[ \square \]

P 1.179. Let \( a, b, c, d \) be positive real numbers such that \( abcd = 1 \). Prove that
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{9}{a + b + c + d} \geq \frac{25}{4}.
\]
Solution (by Vo Quoc Ba Can). Replacing $a, b, c, d$ by $a^4, b^4, c^4, d^4$, respectively, the inequality becomes as follows:

$$
\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} + \frac{9}{a^4 + b^4 + c^4 + d^4} \geq \frac{25}{4abcd},
$$

$$
\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} - \frac{4}{abcd} \geq \frac{9}{a^4 + b^4 + c^4 + d^4},
$$

$$
\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} - \frac{4}{abcd} \geq \frac{9(a^4 + b^4 + c^4 + d^4 - 4abcd)}{4abcd(a^4 + b^4 + c^4 + d^4)}.
$$

Using the identities

$$
a^4 + b^4 + c^4 + d^4 - 4abcd = (a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(ab - cd)^2,
$$

$$
\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} - \frac{4}{abcd} = \frac{(a^2 - b^2)^2}{a^4b^4} + \frac{(c^2 - d^2)^2}{c^4d^4} + \frac{2(ab - cd)^2}{a^2b^2c^2d^2},
$$

the inequality can be written as

$$
\frac{(a^2 - b^2)^2}{a^4b^4} + \frac{(c^2 - d^2)^2}{c^4d^4} + \frac{2(ab - cd)^2}{a^2b^2c^2d^2} \geq \frac{9[(a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(ab - cd)^2]}{4abcd(a^4 + b^4 + c^4 + d^4)},
$$

$$
(a^2 - b^2)^2 \left[ \frac{4cd(a^4 + b^4 + c^4 + d^4)}{a^3b^3} - 9 \right] + (c^2 - d^2)^2 \left[ \frac{4ab(a^4 + b^4 + c^4 + d^4)}{c^3d^3} - 9 \right] + 2(ab - cd)^2 \left[ \frac{4(a^4 + b^4 + c^4 + d^4)}{abcd} - 9 \right] \geq 0.
$$

By the AM-GM inequality, we have

$$
a^4 + b^4 + c^4 + d^4 \geq 4abcd.
$$

Therefore, it suffices to show that

$$
(a^2 - b^2)^2 \left[ \frac{4cd(a^4 + b^4 + c^4 + d^4)}{a^3b^3} - 9 \right] + (c^2 - d^2)^2 \left[ \frac{4ab(a^4 + b^4 + c^4 + d^4)}{c^3d^3} - 9 \right] \geq 0.
$$

Without loss of generality, assume that $a \geq c \geq d \geq b$. Since

$$
(a^2 - b^2)^2 \geq (c^2 - d^2)^2
$$

and

$$
\frac{4cd(a^4 + b^4 + c^4 + d^4)}{a^3b^3} \geq \frac{4(a^4 + b^4 + c^4 + d^4)}{a^3b} \geq \frac{4(a^4 + 3b^4)}{a^3b} > 9,
$$

it is enough to prove that

$$
\left[ \frac{4cd(a^4 + b^4 + c^4 + d^4)}{a^3b^3} - 9 \right] + \left[ \frac{4ab(a^4 + b^4 + c^4 + d^4)}{c^3d^3} - 9 \right] \geq 0,
$$
which is equivalent to
\[
2(a^4 + b^4 + c^4 + d^4) \left( \frac{cd}{a^3b^3} + \frac{ab}{c^3d^3} \right) \geq 9.
\]
Indeed, by the AM-GM inequality,
\[
2(a^4 + b^4 + c^4 + d^4) \left( \frac{cd}{a^3b^3} + \frac{ab}{c^3d^3} \right) \geq 8abcd \frac{2}{abcd} = 16 > 9.
\]
The equality occurs for \(a = b = c = d = 1\). 

\[\Box\]

**P 1.180.** If \(a, b, c, d\) are real numbers such that \(a + b + c + d = 0\), then
\[
\frac{(a - 1)^2}{3a^2 + 1} + \frac{(b - 1)^2}{3b^2 + 1} + \frac{(c - 1)^2}{3c^2 + 1} + \frac{(d - 1)^2}{3d^2 + 1} \leq 4.
\]

**Solution.** Since
\[
4 - 3\frac{(a - 1)^2}{3a^2 + 1} = \frac{(3a + 1)^2}{3a^2 + 1},
\]
we can write the inequality as
\[
\sum \frac{(3a + 1)^2}{3a^2 + 1} \geq 4.
\]
On the other hand, since
\[
4a^2 = 3a^2 + (b + c + d)^2 \leq 3a^2 + 3(b^2 + c^2 + d^2) = 3(a^2 + b^2 + c^2 + d^2),
\]
\[
3a^2 + 1 \leq \frac{9}{4}(a^2 + b^2 + c^2 + d^2) + 1 = \frac{9(a^2 + b^2 + c^2 + d^2) + 4}{4},
\]
we have
\[
\sum \frac{(3a + 1)^2}{3a^2 + 1} \geq \frac{4\sum (3a + 1)^2}{9(a^2 + b^2 + c^2 + d^2) + 4} = 4.
\]
The equality holds for \(a = b = c = d = 0\), and also for \(a = 1\) and \(b = c = d = -1/3\) (or any cyclic permutation).

**Remark.** The following generalization is also true.
- If \(a_1, a_2, \ldots, a_n\) are real numbers such that \(a_1 + a_2 + \cdots + a_n = 0\), then
\[
\frac{(a_1 - 1)^2}{(n - 1)a_1^2 + 1} + \frac{(a_2 - 1)^2}{(n - 1)a_2^2 + 1} + \cdots + \frac{(a_n - 1)^2}{(n - 1)a_n^2 + 1} \leq n,
\]
with equality for \(a_1 = a_2 = \cdots = a_n = 0\), and also for \(a_1 = 1\) and \(a_2 = a_3 = \cdots = a_n = -1/(n - 1)\) (or any cyclic permutation). 

\[\Box\]
If \(a, b, c, d \geq -5\) such that \(a + b + c + d = 4\), then
\[
\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} + \frac{1-d}{(1+d)^2} \geq 0.
\]

**Solution.** Assume that \(a \leq b \leq c \leq d\). We show first that \(x \in [-5, -1) \cup (-1, \infty)\) involves
\[
\frac{1-x}{(1+x)^2} \geq -\frac{1}{8},
\]
and \(x \in [-5, -1) \cup (-1, 1/3]\) involves
\[
\frac{1-x}{(1+x)^2} \geq \frac{3}{8}.
\]
Indeed, we have
\[
\frac{1-x}{(1+x)^2} + \frac{1}{8} = \frac{(x-3)^2}{8(1+x)^2} \geq 0
\]
and
\[
\frac{1-x}{(1+x)^2} - \frac{3}{8} = \frac{(5+x)(1-3x)}{8(1+x)^2} \geq 0.
\]
Then, if \(a \leq 1/3\), then
\[
\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} + \frac{1-d}{(1+d)^2} \geq \frac{3}{8} - \frac{1}{8} - \frac{1}{8} - \frac{1}{8} = 0.
\]
Assume now that \(1/3 \leq a \leq b \leq c \leq d\). Since
\[
1-a \geq 1-b \geq 1-c \geq 1-d
\]
and
\[
\frac{1}{(1+a)^2} \geq \frac{1}{(1+b)^2} \geq \frac{1}{(1+c)^2} \geq \frac{1}{(1+d)^2},
\]
by Chebyshev’s inequality, we have
\[
\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} + \frac{1-d}{(1+d)^2} \geq \frac{1}{4} \left[ \sum (1-a) \right] \left[ \sum \frac{1}{(1+a)^2} \right] = 0.
\]
The equality holds for \(a = b = c = d = 1\), and also for \(a = -5\) and \(b = c = d = 3\) (or any cyclic permutation).
P 1.182. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that

$$\sum \frac{1}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} \leq \frac{1}{2}.$$ 

\textit{(Vasile Cîrtoaje, 2008)}

\textbf{First Solution.} By the Cauchy-Schwarz inequality, we have

$$\sum \frac{n^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} = \sum \frac{(a_1 + a_2 + \cdots + a_n)^2}{2a_1^2 + (a_1 + a_2)^2 + \cdots + (a_1 + a_2 + \cdots + a_n)} \leq \sum \left( \frac{1}{2n} \frac{a_2^2}{a_1^2 + a_2^2} + \cdots + \frac{a_n^2}{a_1^2 + a_n^2} \right)$$

$$= \frac{n}{2} + \frac{n(n-1)}{2} = \frac{n^2}{2},$$

from which the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

\textbf{Second Solution.} Write the inequality as

$$\sum \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} \leq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{2}.$$ 

Since

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} = 1 - \frac{na_1^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2},$$

we need to prove that

$$\sum \frac{a_1^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} + \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{2n} \geq 1.$$ 

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a_1^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{\sum [(n+1)a_1^2 + a_2^2 + \cdots + a_n^2]}$$

$$= \frac{n}{2(a_1^2 + a_2^2 + \cdots + a_n^2)}.$$ 

Then, it suffices to prove that

$$\frac{n}{a_1^2 + a_2^2 + \cdots + a_n^2} + \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \geq 2,$$

which follows immediately from the AM-GM inequality.
P 1.183. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = 0$. Prove that

$$\frac{(a_1 + 1)^2}{a_1^2 + n - 1} + \frac{(a_2 + 1)^2}{a_2^2 + n - 1} + \cdots + \frac{(a_n + 1)^2}{a_n^2 + n - 1} \geq \frac{n}{n - 1}.$$  

(Vasile Cirtoaje, 2010)

**Solution.** Without loss of generality, assume that $a_n^2 = \max\{a_1^2, a_2^2, \cdots, a_n^2\}$. Since

$$\frac{(a_n + 1)^2}{a_n^2 + n - 1} = \frac{n}{n - 1} - \frac{(n - 1 - a_n)^2}{(n - 1)(a_n^2 + n - 1)},$$

we can write the inequality as

$$\sum_{i=1}^{n-1} \frac{(a_i + 1)^2}{a_i^2 + n - 1} \geq \frac{(n - 1 - a_n)^2}{(n - 1)(a_n^2 + n - 1)}.$$

From the Cauchy-Schwarz inequality

$$\left[ \sum_{i=1}^{n-1} (a_i^2 + n - 1) \right] \left[ \sum_{i=1}^{n-1} \frac{(a_i + 1)^2}{a_i^2 + n - 1} \right] \geq \left[ \sum_{i=1}^{n-1} (a_i + 1) \right]^2,$$

we get

$$\sum_{i=1}^{n-1} \frac{(a_i + 1)^2}{a_i^2 + n - 1} \geq \frac{(n - 1 - a_n)^2}{\sum_{i=1}^{n-1} a_i^2 + (n - 1)^2}.$$

Thus, it suffices to show that

$$\sum_{i=1}^{n-1} a_i^2 + (n - 1)^2 \leq (n - 1)(a_n^2 + n - 1),$$

which is clearly true. The proof is completed. The equality holds for $\frac{-a_1}{n - 1} = a_2 = a_3 = \cdots = a_n$ (or any cyclic permutation).

\[\square\]

P 1.184. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that

$$\frac{1}{1 + (n - 1)a_1} + \frac{1}{1 + (n - 1)a_2} + \cdots + \frac{1}{1 + (n - 1)a_n} \geq 1.$$  

(Vasile Cirtoaje, 1991)
First Solution. Let $k = (n - 1)/n$. We can get the required inequality by summing the inequalities below for $i = 1, 2, \cdots, n$:

$$\frac{1}{1 + (n - 1)a_i} \geq \frac{a_i^{-k}}{a_1^{-k} + a_2^{-k} + \cdots + a_n^{-k}}.$$

This inequality is equivalent to

$$a_1^{-k} + \cdots + a_i^{-k} + a_{i+1}^{-k} + \cdots + a_n^{-k} \geq (n - 1)a_i^{-k},$$

which follows from the AM-GM inequality. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. Using the substitutions $a_i = 1/x_i$ for all $i$, the inequality becomes

$$\frac{x_1}{x_1 + n - 1} + \frac{x_2}{x_2 + n - 1} + \cdots + \frac{x_n}{x_n + n - 1} \geq 1,$$

where $x_1, x_2, \cdots, x_n$ are positive real numbers such that $x_1 x_2 \cdots x_n = 1$. By the Cauchy-Schwarz inequality, we have

$$\sum x_i / (x_i + n - 1) \geq (\sum \sqrt{x_i})^2 / \sum (x_i + n - 1).$$

Thus, we still have to prove that

$$(\sum \sqrt{x_i})^2 \geq \sum x_1 + n(n - 1),$$

which reduces to

$$\sum_{1 \leq i < j \leq n} \sqrt{x_i x_j} \geq \frac{n(n - 1)}{2}.$$

Since $x_1 x_2 \cdots x_n = 1$, this inequality follows from the AM-GM inequality.

Third Solution. For the sake of contradiction, assume that the required inequality is not true. Then, it suffices to show that the hypothesis $a_1 a_2 \cdots a_n = 1$ does not hold. More precisely, we will prove that

$$\frac{1}{1 + (n - 1)a_1} + \frac{1}{1 + (n - 1)a_2} + \cdots + \frac{1}{1 + (n - 1)a_n} < 1$$

involves $a_1 a_2 \cdots a_n > 1$. Let $x_i = \frac{1}{1 + (n - 1)a_i}$, $0 < x_i < 1$, for $i = 1, 2, \cdots, n$. Since $a_i = 1 - x_i (n - 1)x_i$ for all $i$, we need to show that

$$x_1 + x_2 + \cdots + x_n < 1.$$
implies
\[(1 - x_1)(1 - x_2) \cdots (1 - x_n) > (n - 1)^n x_1 x_2 \cdots x_n.\]

Using the AM-GM inequality, we have
\[1 - x_i > \sum_{k \neq i} x_k \geq (n - 1) x_i \sqrt[n-1]{\prod_{k \neq i} x_k}.\]

Multiplying the inequalities
\[1 - x_i > (n - 1) x_i \sqrt[n-1]{\prod_{k \neq i} x_k}.\]

for \(i = 1, 2, \cdots, n\), the conclusion follows.

**Remark.** The inequality in P 1.184 is a particular case of the following more general results (Vasile Cîrtoaje, 2005):

- Let \(a_1, a_2, \ldots, a_n\) be positive real numbers such that \(a_1 a_2 \cdots a_n = 1\). If \(0 < k \leq n - 1\) and \(p \geq n^{1/k} - 1\), then
  \[\frac{1}{(1 + pa_1)^k} + \frac{1}{(1 + pa_2)^k} + \cdots + \frac{1}{(1 + pa_n)^k} \geq \frac{n}{(1 + p)^k}.\]

- Let \(a_1, a_2, \ldots, a_n\) be positive real numbers such that \(a_1 a_2 \cdots a_n = 1\). If \(k \geq \frac{1}{n - 1}\) and
  \(0 < p \leq \left(\frac{n}{n - 1}\right)^{1/k} - 1\), then
  \[\frac{1}{(1 + pa_1)^k} + \frac{1}{(1 + pa_2)^k} + \cdots + \frac{1}{(1 + pa_n)^k} \leq \frac{n}{(1 + p)^k}.\]

\(\Box\)

**P 1.185.** Let \(a_1, a_2, \ldots, a_n\) be positive real numbers such that \(a_1 a_2 \cdots a_n = 1\). Prove that
\[\frac{1}{1 - a_1 + na_1^2} + \frac{1}{1 - a_2 + na_2^2} + \cdots + \frac{1}{1 - a_n + na_n^2} \geq 1.\]

**(Vasile Cîrtoaje, 2009)**

**Solution.** First, we show that
\[\frac{1}{1 - x + nx^2} \geq \frac{1}{1 + (n - 1)x^k},\]
where \( x > 0 \) and \( k = 2 + \frac{1}{n-1} \). Write the inequality as
\[
(n-1)x^k + x \geq nx^2.
\]
We can get this inequality using the AM-GM inequality as follows
\[
(n-1)x^k + x \geq n\sqrt[k]{x^{(n-1)k}x} = nx^2.
\]
Thus, it suffices to show that
\[
\frac{1}{1 + (n-1)a_1^k} + \frac{1}{1 + (n-1)a_2^k} + \cdots + \frac{1}{1 + (n-1)a_n^k} \geq 1,
\]
which follows immediately from the inequality in the preceding P 1.184. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

**Remark 1.** Similarly, we can prove the following more general statement.

- Let \( a_1, a_2, \ldots, a_n \) be positive real numbers such that \( a_1a_2\cdots a_n = 1 \). If \( p \) and \( q \) are real numbers such that \( p + q = n - 1 \) and \( n - 1 \leq q \leq (\sqrt{n} + 1)^2 \), then
\[
\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq 1.
\]

**Remark 2.** We can extend the inequality in Remark 1 as follows (Vasile Cîrtoaje, 2009).

- Let \( a_1, a_2, \ldots, a_n \) be positive real numbers such that \( a_1a_2\cdots a_n = 1 \). If \( p \) and \( q \) are real numbers such that \( p + q = n - 1 \) and \( 0 \leq q \leq (\sqrt{n} + 1)^2 \), then
\[
\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq 1.
\]

\( \square \)

**P 1.186.** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers such that
\[
a_1, a_2, \ldots, a_n \geq \frac{k(n-k-1)}{kn-k-1}, \quad k > 1
\]
and
\[
a_1a_2\cdots a_n = 1.
\]
Prove that
\[
\frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k} \leq \frac{n}{1+k}.
\]

(Vasile Cîrtoaje, 2005)
**Solution.** We use the induction on \( n \). Let

\[
E_n(a_1, a_2, \ldots, a_n) = \frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k} - \frac{n}{1 + k}.
\]

For \( n = 2 \), we have

\[
E_2(a_1, a_2) = \frac{(1-k)(\sqrt{a_1} - \sqrt{a_2})^2}{(1+k)(a_1+k)(a_2+k)} \leq 0.
\]

Assume that the inequality is true for \( n-1 \) numbers \( (n \geq 3) \), and prove that \( E_n(a_1, a_2, \ldots, a_n) \geq 0 \) for \( a_1 a_2 \cdots a_n = 1 \) and \( a_1, a_2, \ldots, a_n \geq p_n \), where

\[
p_n = \frac{k(n-k-1)}{kn-k-1}.
\]

Due to symmetry, we may assume that \( a_1 \geq 1 \) and \( a_2 \leq 1 \). There are two cases to consider.

**Case 1:** \( a_1 a_2 \leq k^2 \). Since \( a_1 a_2 \geq a_2 \) and \( p_{n-1} < p_n \), from \( a_1, a_2, \ldots, a_n \geq p_n \) it follows that

\[
a_1 a_2, a_3, \ldots, a_n > p_{n-1}.
\]

Then, by the inductive hypothesis, we have \( E_{n-1}(a_1 a_2, a_3, \ldots, a_n) \leq 0 \), and it suffices to show that

\[
E_n(a_1, a_2, \ldots, a_n) \leq E_{n-1}(a_1 a_2, a_3, \ldots, a_n).
\]

This is equivalent to

\[
\frac{1}{a_1 + k} + \frac{1}{a_2 + k} - \frac{1}{a_1 a_2 + k} - \frac{1}{1 + k} \leq 0,
\]

which reduces to the obvious inequality

\[
(a_1 - 1)(1 - a_2)(a_1 a_2 - k^2) \leq 0.
\]

**Case 2:** \( a_1 a_2 \geq k^2 \). Since

\[
\frac{1}{a_1 + k} + \frac{1}{a_2 + k} = \frac{a_1 + a_2 + 2k}{a_1 a_2 + k(a_1 + a_2) + k^2} \leq \frac{a_1 + a_2 + 2k}{k^2 + k(a_1 + a_2) + k^2} = \frac{1}{k}
\]

and

\[
\frac{1}{a_3 + k} + \cdots + \frac{1}{a_n + k} \leq \frac{n-2}{p_n + k} = \frac{kn - k - 1}{k(k+1)}
\]

we have

\[
E_n(a_1, a_2, \ldots, a_n) \leq \frac{1}{k} + \frac{kn - k - 1}{k(k+1)} - \frac{n}{1 + k} = 0.
\]

Thus, the proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).
**Remark.** For \( k = n - 1 \), we get the inequality in P 1.184. Also, for \( k \to \infty \), we get the known inequalities

\[
1 < \frac{1}{1 + a_1} + \frac{1}{1 + a_2} + \cdots + \frac{1}{1 + a_n} < n - 1,
\]

which holds for all positive numbers \( a_1, a_2, \ldots, a_n \) satisfying \( a_1 a_2 \cdots a_n = 1 \).

\[\square\]

**P 1.187.** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers such that

\[
a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \quad a_1 a_2 \cdots a_n = 1.
\]

Prove that

\[
\frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2} \geq 0.
\]

(Vasile Cirtoaje, 2013)

**Solution.** For \( n = 2 \), we have

\[
\frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} = \frac{(a_1 - 1)^4}{(3 + a_1^2)(3a_1^2 + 1)} \geq 0.
\]

For \( n \geq 3 \), we will use the induction method and the inequality

\[
\frac{1 - x}{3 + x^2} + \frac{1 - y}{3 + y^2} \geq \frac{1 - xy}{3 + x^2y^2},
\]

which holds for all \( x, y \in [0, 1] \). Indeed, we can write this inequality as

\[
(1 - x)(1 - y)(3 + xy)(3 - xy - x^2y - xy^2) \geq 0,
\]

which is obviously true. Based on the inequality

\[
\frac{1 - a_{n-1}}{3 + a_{n-1}^2} + \frac{1 - a_n}{3 + a_n^2} \geq \frac{1 - a_{n-1}a_n}{3 + a_{n-1}^2a_n^2},
\]

it suffices to show that

\[
\frac{1 - a_1}{3 + a_1^2} + \cdots + \frac{1 - a_{n-2}}{3 + a_{n-2}^2} + \frac{1 - a_{n-1}a_n}{3 + a_{n-1}^2a_n^2} \geq 0.
\]

Since

\[
a_1 \geq 1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n,
\]

this inequality follows from the hypothesis induction. Thus, the proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

\[\square\]
Symmetric Rational Inequalities

P 1.188. If \( a_1, a_2, \ldots, a_n \geq 0 \), then
\[
\frac{1}{1 + na_1} + \frac{1}{1 + na_2} + \cdots + \frac{1}{1 + na_n} \geq \frac{n}{n + a_1 a_2 \cdots a_n}.
\]

(Vasile Cîrtoaje, 2013)

**Solution.** If one of \( a_1, a_2, \ldots, a_n \) is zero, the inequality is obvious. Consider further that \( a_1, a_2, \ldots, a_n > 0 \) and let
\[
r = \sqrt[1+n]{a_1 a_2 \cdots a_n}.
\]

By the Cauchy-Schwarz inequality, we have
\[
\sum \frac{1}{1 + na_1} \geq \frac{(\sum \sqrt[a_2 a_3 \cdots a_n]{})^2}{\sum (1 + na_1) a_2 a_3 \cdots a_n} = \frac{(\sum \sqrt[a_2 a_3 \cdots a_n]{})^2}{\sum a_2 a_3 \cdots a_n + n^2 r^n}.
\]

Therefore, it suffices to show that
\[
(n + r^n)(\sum \sqrt[a_2 a_3 \cdots a_n]{})^2 \geq n \sum a_2 a_3 \cdots a_n + n^3 r^n.
\]

By the AM-GM inequality, we have
\[
(\sum \sqrt[a_2 a_3 \cdots a_n]{})^2 \geq \sum a_2 a_3 \cdots a_n + n(n-1)r^{n-1}.
\]

Thus, it is enough to prove that
\[
(n + r^n)\left[\sum a_2 a_3 \cdots a_n + n(n-1)r^{n-1}\right] \geq n \sum a_2 a_3 \cdots a_n + n^3 r^n,
\]

which is equivalent to
\[
r^n \sum a_2 a_3 \cdots a_n + n(n-1)r^{2n-1} + n^2(n-1)r^{n-1} \geq n^3 r^n.
\]

Also, by the AM-GM inequality,
\[
\sum a_2 a_3 \cdots a_n \geq nr^{n-1},
\]
and it suffices to show the inequality
\[
.nr^{2n-1} + n(n-1)r^{2n-1} + n^2(n-1)r^{n-1} \geq n^3 r^n,
\]

which can be rewritten as
\[
n^2 r^{n-1}(r^n - nr + n - 1) \geq 0.
\]

Indeed, by the AM-GM inequality, we get
\[
r^n - nr + n - 1 = r^n + 1 + \cdots + 1 - nr \geq n^{\sqrt[n]{r^n \cdot 1 \cdots 1 - nr}} = 0.
\]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).
P 1.189. If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \geq \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n},$$

where

$$b_i = \frac{1}{n-1} \sum_{j \neq i} a_j, \quad i = 1, 2, \ldots, n.$$

**Solution.** Let

$$a = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

$$A = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

Using the Cauchy-Schwarz inequality, we have

$$\frac{(n-1)^2}{a_2 + a_3 + \cdots + a_n} \leq \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} = A - \frac{1}{a_1},$$

$$\frac{n-1}{b_1} \leq A - \frac{1}{a_1},$$

$$\frac{a_1}{b_1} \leq \frac{Aa_1 - 1}{n-1},$$

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \leq \frac{A}{n-1} - \frac{n}{n-1},$$

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \leq \frac{naA}{n-1} - \frac{n}{n-1}.$$

Since

$$\sum_{i=1}^{n} \frac{b_i}{a_i} = \frac{1}{n-1} \sum_{i=1}^{n} \frac{na - a_i}{a_i} = \frac{naA}{n-1} - \frac{n}{n-1},$$

the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_n$. \qed

---

P 1.190. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

then
Symmetric Rational Inequalities

(a) \[
\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} \geq 1; \\
\]

(b) \[
\frac{1}{n - 1 + a_1} + \frac{1}{n - 1 + a_2} + \cdots + \frac{1}{n - 1 + a_n} \leq 1.
\]

(Vasile Cîrtoaje, 1996)

Solution. (a) We use the contradiction method. So, assume that
\[
\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} < 1,
\]
and show that
\[
a_1 + a_2 + \cdots + a_n > \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.
\]
Using the substitution
\[
x_i = \frac{1}{1 + (n-1)a_i}, \quad i = 1, 2, \ldots, n,
\]
the hypothesis inequality becomes
\[
x_1 + x_2 + \cdots + x_n < 1.
\]
This inequality involves
\[
1 - x_i > (n-1)b_i, \quad b_i = \frac{1}{n-1} \sum_{j \neq i} x_j, \quad i = 1, 2, \ldots, n.
\]
Using this result and the inequality from the preceding P 1.189, we get
\[
a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} \frac{1-x_i}{(n-1)x_i} > \sum_{i=1}^{n} \frac{b_i}{x_i} \geq \sum_{i=1}^{n} \frac{x_i}{b_i}.
\]
Thus, it suffices to show that
\[
\sum_{i=1}^{n} \frac{x_i}{b_i} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.
\]
Indeed, we have
\[
\sum_{i=1}^{n} \frac{x_i}{b_i} > \sum_{i=1}^{n} \frac{(n-1)x_i}{1-x_i} = \sum_{i=1}^{n} \frac{1}{a_i}.
\]
The proof is completed. The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\).

(b) The desired inequality follows from the inequality in (a) by replacing \(a_1, a_2, \ldots, a_n\) with \(1/a_1, 1/a_2, \ldots, 1/a_n\), respectively.

\[\square\]
Appendix A

Glossary

1. **AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY**
   If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then
   \[ a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 a_2 \cdots a_n}, \]
   with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. **WEIGHTED AM-GM INEQUALITY**
   Let $p_1, p_2, \cdots, p_n$ be positive real numbers satisfying
   \[ p_1 + p_2 + \cdots + p_n = 1. \]
   If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then
   \[ p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}, \]
   with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. **AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY**
   If $a_1, a_2, \cdots, a_n$ are positive real numbers, then
   \[ (a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq n^2, \]
   with equality if and only if $a_1 = a_2 = \cdots = a_n$.
4. POWER MEAN INEQUALITY

The power mean of order $k$ of positive real numbers $a_1, a_2, \cdots, a_n$, that is

$$M_k = \begin{cases} \left( \frac{a_1^k + a_2^k + \cdots + a_n^k}{n} \right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instance, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

5. BERNOULLI’S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1 + x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1 + x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

In addition, if $a_1, a_2, \cdots, a_n$ are real numbers such that either $a_1, a_2, \cdots, a_n \geq 0$ or $-1 \leq a_1, a_2, \cdots, a_n \leq 0$, then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR’S INEQUALITY

For any nonnegative real numbers $a, b, c$ and any positive number $k$, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

For $k = 1$, we get the third degree Schur’s inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(b + c) + ca(c + a),$$

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a + b + c} \geq 2(ab + bc + ca),$$

$$(b - c)^2(b + c - a) + (c - a)^2(c + a - b) + (a - b)^2(a + b - c) \geq 0.$$
For \( k = 2 \), we get the fourth degree Schur’s inequality, which holds for any real numbers \( a, b, c \), and can be rewritten as follows

\[
a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2),
\]

\[
(b - c)^2(b + c - a)^2 + (c - a)^2(c + a - b)^2 + (a - b)^2(a + b - c)^2 \geq 0,
\]

\[
6abc \geq (p^2 - q)(4q - p^2),
\]

where \( p = a + b + c \), \( q = ab + bc + ca \).

A generalization of the fourth degree Schur’s inequality, which holds for any real numbers \( a, b, c \) and any real number \( m \), is the following (Vasile Cirtoaje, 2004):

\[
\sum (a - mb)(a - mc)(a - b)(a - c) \geq 0,
\]

where the equality holds for \( a = b = c \), and for \( a/m = b = c \) (or any cyclic permutation).

This inequality is equivalent to

\[
\sum a^4 + m(m + 2) \sum a^2b^2 + (1 - m^2)abc \sum a \geq (m + 1) \sum ab(a^2 + b^2),
\]

\[
\sum (b - c)^2(b + c - a - ma)^2 \geq 0.
\]

Another generalization of the fourth degree Schur’s inequality (Vasile Cirtoaje, 2004):

Let \( \alpha, \beta, \gamma \) be real numbers such that

\[1 + \alpha + \beta = 2\gamma.\]

The inequality

\[
\sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a \geq \gamma \sum ab(a^2 + b^2)
\]

holds for any real numbers \( a, b, c \) if and only if

\[1 + \alpha \geq \gamma^2.\]

\[\square\]

7. CAUCHY-SCHWARZ INEQUALITY

For any real numbers \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) we have

\[
(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,
\]

with equality if and only if \( a_i \) and \( b_i \) are proportional for all \( i \).

\[\square\]
8. HÖLDER’S INEQUALITY

If \( x_{ij} (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n) \) are nonnegative real numbers, then
\[
\prod_{i=1}^{m} \left( \sum_{j=1}^{n} x_{ij} \right)^{\frac{m}{n}} \leq \left( \sum_{i=1}^{n} \prod_{i=1}^{m} x_{ij} \right)^{\frac{m}{n}}.
\]

\( \square \)

9. CHEBYSHEV’S INEQUALITY

Let \( a_1 \geq a_2 \geq \cdots \geq a_n \) be real numbers.

a) If \( b_1 \geq b_2 \geq \cdots \geq b_n \), then
\[
n \sum_{i=1}^{n} a_i b_i \geq \left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} b_i \right);
\]

b) If \( b_1 \leq b_2 \leq \cdots \leq b_n \), then
\[
n \sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} b_i \right).
\]

\( \square \)

10. MINKOWSKI’S INEQUALITY

For any real number \( k \geq 1 \) and any positive real numbers \( a_1, a_2, \cdots, a_n \) and \( b_1, b_2, \cdots, b_n \), the inequalities hold
\[
\sum_{i=1}^{n} \left( a_i^k + b_i^k \right)^{\frac{1}{k}} \geq \left( \sum_{i=1}^{n} a_i \right)^k + \left( \sum_{i=1}^{n} b_i \right)^k \]
\[
\sum_{i=1}^{n} (a_i^k + b_i^k + c_i^k)^{\frac{1}{k}} \geq \left( \sum_{i=1}^{n} a_i \right)^k + \left( \sum_{i=1}^{n} b_i \right)^k + \left( \sum_{i=1}^{n} c_i \right)^k \]

\( \square \)

11. REARRANGEMENT INEQUALITY

(1) If \( a_1, a_2, \cdots, a_n \) and \( b_1, b_2, \cdots, b_n \) are two increasing (or decreasing) real sequences, and \((i_1, i_2, \cdots, i_n)\) is an arbitrary permutation of \((1, 2, \cdots, n)\), then
\[
a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}
\]
and
\[ n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n). \]

(2) If \( a_1, a_2, \cdots, a_n \) is decreasing and \( b_1, b_2, \cdots, b_n \) is increasing, then
\[ a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n} \]
and
\[ n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \leq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n). \]

(3) Let \( b_1, b_2, \cdots, b_n \) and \( c_1, c_2, \cdots, c_n \) be two real sequences such that
\[ b_1 + \cdots + b_k \geq c_1 + \cdots + c_k, \quad k = 1, 2, \cdots, n. \]
If \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \), then
\[ a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 c_1 + a_2 c_2 + \cdots + a_n c_n. \]

Notice that all these inequalities follow immediately from the identity
\[ \sum_{i=1}^{n} a_i (b_i - c_i) = \sum_{i=1}^{n} (a_i - a_{i+1}) \left( \sum_{j=1}^{i} b_j - \sum_{j=1}^{i} c_j \right), \]
where \( a_{n+1} = 0. \)

\[ \square \]

12. MACLAURIN’S INEQUALITY and NEWTON’S INEQUALITY

If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers, then
\[ S_1 \geq S_2 \geq \cdots \geq S_n \quad (Maclaurin) \]
and
\[ S_k^2 \geq S_{k-1} S_{k+1}, \quad (Newton) \]
where
\[ S_k = \sqrt[\binom{n}{k}]{\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} a_{i_2} \cdots a_{i_k}}. \]

\[ \square \]
13. CONVEX FUNCTIONS

A function \( f \) defined on a real interval \( I \) is said to be convex if

\[
f(ax + \beta y) \leq \alpha f(x) + \beta f(y)
\]

for all \( x, y \in I \) and any \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \). If the inequality is reversed, then \( f \) is said to be concave.

If \( f \) is differentiable on \( I \), then \( f \) is (strictly) convex if and only if the derivative \( f' \) is (strictly) increasing. If \( f'' \geq 0 \) on \( I \), then \( f \) is convex on \( I \). Also, if \( f'' \geq 0 \) on \( (a, b) \) and \( f \) is continuous on \( [a, b] \), then \( f \) is convex on \( [a, b] \).

A function \( f : I \rightarrow \mathbb{R} \) is half convex on a real interval \( I \) if there exists a point \( s \in I \) such that \( f \) is convex on \( I_{u \leq s} \) or \( I_{u \geq s} \).

A function \( f : I \rightarrow \mathbb{R} \) is right partially convex related to a a point \( s \in I \) if there exists a number \( s_0 \in I, s_0 > s \), such that \( f \) is convex on \( I_{u \in [s, s_0]} \). Also, a function \( f : I \rightarrow \mathbb{R} \) is left partially convex related to a a point \( s \in I \) if there exists a point \( s_0 \in I, s_0 < s \), such that \( f \) is convex on \( I_{u \in [s_0, s]} \).

**Jensen’s inequality.** Let \( p_1, p_2, \ldots, p_n \) be positive real numbers. If \( f \) is a convex function on a real interval \( I \), then for any \( a_1, a_2, \ldots, a_n \in I \), the inequality holds

\[
\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \geq f \left( \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n} \right).
\]

For \( p_1 = p_2 = \cdots = p_n \), Jensen’s inequality becomes

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right).
\]

Based on the following three theorems, we can extend this form of Jensen’s inequality to half or partially convex functions.

**Half Convex Function-Theorem** (Vasile Cirtoaje, 2004). Let \( f(u) \) be a function defined on a real interval \( I \) and convex on \( I_{u \leq s} \) or \( I_{u \geq s} \), where \( s \in I \). The inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)
\]

holds for all \( a_1, a_2, \ldots, a_n \in I \) satisfying \( a_1 + a_2 + \cdots + a_n = ns \) if and only if

\[
f(x) + (n - 1)f(y) \geq nf(s)
\]

for all \( x, y \in I \) such that \( x + (n - 1)y = ns \).

**Right Partially Convex Function-Theorem** (Vasile Cirtoaje, 2012). Let \( f \) be a function defined on a real interval \( I \) and convex on \( [s, s_0] \), where \( s, s_0 \in I, s < s_0 \). In addition, \( f \) is decreasing on \( I_{u \leq s_0} \) and

\[
\min_{u \geq s} f(u) = f(s_0).
\]
The inequality

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \]

holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying \( a_1 + a_2 + \cdots + a_n = ns \) if and only if

\[ f(x) + (n-1)f(y) \geq nf(s) \]

for all \( x, y \in \mathbb{I} \) such that \( x \leq s \leq y \) and \( x + (n-1)y = ns \).

**Left Partially Convex Function-Theorem** (Vasile Cirtoaje, 2012). Let \( f \) be a function defined on a real interval \( \mathbb{I} \) and convex on \([s_0, s]\), where \( s_0, s \in \mathbb{I}, s_0 < s \). In addition, \( f \) is increasing on \( I_{u \geq s_0} \) and satisfies

\[ \min_{u \leq s} f(u) = f(s_0). \]

The inequality

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \]

holds for all \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) satisfying \( a_1 + a_2 + \cdots + a_n = ns \) if and only if

\[ f(x) + (n-1)f(y) \geq nf(s) \]

for all \( x, y \in \mathbb{I} \) such that \( x \geq s \geq y \) and \( x + (n-1)y = ns \).

In all these theorems, we may replace the hypothesis condition

\[ f(x) + (n-1)f(y) \geq nf(s), \]

by the equivalent condition

\[ h(x, y) \geq 0 \quad \text{for all} \quad x, y \in \mathbb{I} \quad \text{such that} \quad x + (n-1)y = ns, \]

where

\[ h(x, y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(s)}{u-s}. \]

The following theorem is also useful to prove some symmetric inequalities.

**Left Convex-Right Concave Function Theorem** (Vasile Cirtoaje, 2004). Let \( a < c \) be real numbers, let \( f \) be a continuous function on \( \mathbb{I} = [a, \infty) \), strictly convex on \([a, c]\) and strictly concave on \([c, \infty)\), and let

\[ E(a_1, a_2, \ldots, a_n) = f(a_1) + f(a_2) + \cdots + f(a_n). \]

If \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) such that

\[ a_1 + a_2 + \cdots + a_n = S = \text{constant}, \]


then
(a) \( E \) is minimum for \( a_1 = a_2 = \cdots = a_{n-1} \leq a_n \);
(b) \( E \) is maximum for either \( a_1 = a \) or \( a < a_1 \leq a_2 = \cdots = a_n \).

On the other hand, it is known the following result concerning the best upper bound of Jensen’s difference.

**Best Upper Bound of Jensen’s Difference-Theorem** (Vasile Cirtoaje, 1989). Let \( p_1, p_2, \cdots, p_n \) be fixed positive real numbers, and let \( f \) be a convex function on a closed interval \( \mathbb{I} = [a, b] \). If \( a_1, a_2, \cdots, a_n \in \mathbb{I} \), then Jensen’s difference

\[
D = \frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} - f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right)
\]

is maximum when some of \( a_i \) are equal to \( a \), and the others \( a_i \) are equal to \( b \); that is, when all \( a_i \in \{a, b\} \).

\[\square\]

**14. KARAMATA’S MAJORIZATION INEQUALITY**

We say that a vector \( \vec{A} = (a_1, a_2, \ldots, a_n) \) with \( a_1 \geq a_2 \geq \cdots \geq a_n \) majorizes a vector \( \vec{B} = (b_1, b_2, \ldots, b_n) \) with \( b_1 \geq b_2 \geq \cdots \geq b_n \), and write it as

\[
\vec{A} \geq \vec{B},
\]

if
\[
\begin{align*}
a_1 &\geq b_1, \\
a_1 + a_2 &\geq b_1 + b_2, \\
&\cdots \cdots \\
a_1 + a_2 + \cdots + a_{n-1} &\geq b_1 + b_2 + \cdots + b_{n-1}, \\
a_1 + a_2 + \cdots + a_n &\geq b_1 + b_2 + \cdots + b_n,
\end{align*}
\]

Let \( f \) be a convex function on a real interval \( \mathbb{I} \). If a decreasingly ordered vector

\[
\vec{A} = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},
\]

majorizes a decreasingly ordered vector

\[
\vec{B} = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},
\]

then

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n).
\]

\[\square\]
15. POPOVICIU’S INEQUALITY

If \( f \) is a convex function on a real interval \( \mathbb{I} \) and \( a_1, a_2, \ldots, a_n \in \mathbb{I} \), then
\[
f(a_1) + f(a_2) + \cdots + f(a_n) + n(n-2)f \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \geq \]
\[
(n-1)\left[ f(b_1) + f(b_2) + \cdots + f(b_n) \right],
\]
where
\[
b_i = \frac{1}{n-1} \sum_{j \neq i} a_j, \quad i = 1, 2, \ldots, n.\]

\[\Box\]

16. SQUARE PRODUCT INEQUALITY

Let \( a, b, c \) be real numbers, and let
\[
p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,
\]
\[
s = \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.
\]
From the identity
\[
27(a - b)^2(b - c)^2(c - a)^2 = 4(p^2 - 3q)^3 - (2p^3 - 9pq + 27r)^2,
\]
it follows that
\[
\frac{-2p^3 + 9pq - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27} \leq r \leq \frac{-2p^3 + 9pq + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27},
\]
which is equivalent to
\[
\frac{p^3 - 3ps^2 - 2s^3}{27} \leq r \leq \frac{p^3 - 3ps^2 + 2s^3}{27}.
\]
Therefore, for constant \( p \) and \( q \), the product \( r \) is minimal and maximal when two of \( a, b, c \) are equal.

\[\Box\]

17. VASC’S INEQUALITY

If \( a, b, c \) are real numbers, then
\[
(a^2 + b^2 + c^2)^2 \geq 3(a^3 b + b^3 c + c^3 a),
\]
with equality for \( a = b = c \), and also for
\[
\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}\]
(or any cyclic permutation) - Vasile Cirtoaje, 1991.

A generalization of this inequality is the following (Vasile Cirtoaje, 2007):

$$\sum a^4 + A \sum a^2 b^2 + B a b c \sum a \geq C \sum a^3 b + D \sum a b^3,$$

where $A, B, C, D$ are real numbers such that

$$1 + A + B = C + D,$$

$$3(1 + A) \geq C^2 + CD + D^2.$$

\[\□\]\[\□\]

18. SYMMETRIC INEQUALITIES OF DEGREE THREE, FOUR OR FIVE

Let $f_n(a, b, c)$ be a symmetric homogeneous polynomial of degree $n = 3, 4$ or $5$.

(a) The inequality $f_4(a, b, c) \geq 0$ holds for all real numbers $a, b, c$ if and only if $f_4(a, 1, 1) \geq 0$ for all real $a$;

(b) The inequality $f_6(a, b, c) \geq 0$ holds for all $a, b, c \geq 0$ if and only if $f_6(a, 1, 1) \geq 0$ and $f_6(0, b, c) \geq 0$ for all $a, b, c \geq 0$ (Vasile Cirtoaje, 2008).

\[\□\]\[\□\]

19. SYMMETRIC INEQUALITIES OF DEGREE SIX

Any sixth degree symmetric homogeneous polynomial $f_6(a, b, c)$ can be written in the form

$$f_6(a, b, c) = A r^2 + B(p, q)r + C(p, q),$$

where $A$ is called the highest coefficient of $f_6$, and

$$p = a + b + c, \quad q = a b + b c + c a, \quad r = a b c.$$

Case 1: $A \leq 0$. The following statement holds.

(a) The inequality $f_6(a, b, c) \geq 0$ holds for all real numbers $a, b, c$ if and only if $f_6(a, 1, 1) \geq 0$ for all real $a$;

(b) The inequality $f_6(a, b, c) \geq 0$ holds for all $a, b, c \geq 0$ if and only if $f_6(a, 1, 1) \geq 0$ and $f_6(0, b, c) \geq 0$ for all $a, b, c \geq 0$ (Vasile Cirtoaje, 2008).

Case 2: $A > 0$. We can use the highest coefficient cancellation method (Vasile Cirtoaje, 2008). This method consists in finding some suitable real numbers $B, C$ and $D$ such that the following sharper inequality holds

$$f_6(a, b, c) \geq A \left( r + B p^3 + C p q + D \frac{q^2}{p} \right)^2.$$
Because the function $g_6$ defined by

$$g_6(a, b, c) = f_6(a, b, c) - A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2$$

has the highest coefficient $A_1 = 0$, we can prove the inequality $g_6(a, b, c) \geq 0$ as in the preceding case 1.

Notice that sometimes it is useful to break the problem into two parts, $p^2 \leq \xi q$ and $p^2 > \xi q$, where $\xi$ is a suitable real number.

\[ \square \]

20. EQUAL VARIABLE METHOD

The Equal Variable Theorem (EV-Theorem) for nonnegative real variables has the following statement (Vasile Cirtoaje, 2005).

**EV-Theorem** (for nonnegative variables). Let $a_1, a_2, \cdots, a_n$ ($n \geq 3$) be fixed nonnegative real numbers, and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be nonnegative real variables such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$

$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where $k$ is a real number (for $k = 0$, assume that $x_1x_2\cdots x_n = a_1a_2\cdots a_n > 0$). Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = f'(x^{\frac{k}{k-1}})$$

is strictly convex, and let

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n).$$

1. If $k \leq 0$, then $S_n$ is maximum for

$$0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimum for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n;$$

2. If $k > 0$ and either $f$ is continuous at $x = 0$ or $f(0^+) = -\infty$, then $S_n$ is maximum for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimum for

$$x_1 = \cdots = x_{j-1} = 0, \ x_{j+1} = \cdots = x_n,$$
where \( j \in \{1, 2, \cdots, n\} \).

For \( f(x) = x^m \), we get the following corollary.

**EV-COROLLARY** (for nonnegative variables). Let \( a_1, a_2, \cdots, a_n \ (n \geq 3) \) be fixed nonnegative real numbers, let \( x_1 \leq x_2 \leq \cdots \leq x_n \) be nonnegative real variables such that

\[
x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \\
x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,
\]

and let

\[
S_n = x_1^m + x_2^m + \cdots + x_n^m.
\]

**Case 1**: \( k \leq 0 \) (for \( k = 0 \), assume that \( x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n > 0 \).

(a) If \( m \in (k, 0) \cup (1, \infty) \), then \( S_n \) is maximum for

\[
0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]

and is minimum for

\[
0 < x_1 \leq x_2 = x_3 = \cdots = x_n;
\]

(b) If \( m \in (-\infty, k) \cup (0, 1) \), then \( S_n \) is minimum for

\[
0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]

and is maximum for

\[
0 < x_1 \leq x_2 = x_3 = \cdots = x_n.
\]

**Case 2**: \( 0 < k < 1 \).

(a) If \( m \in (0, k) \cup (1, \infty) \), then \( S_n \) is maximum for

\[
0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]

and is minimum for

\[
x_1 = \cdots = x_{j-1} = 0, \ x_{j+1} = \cdots = x_n,
\]

where \( j \in \{1, 2, \cdots, n\} \);

(b) If \( m \in (-\infty, 0) \cup (k, 1) \), then \( S_n \) is minimum for

\[
0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]

and is maximum for

\[
x_1 = \cdots = x_{j-1} = 0, \ x_{j+1} = \cdots = x_n,
\]

where \( j \in \{1, 2, \cdots, n\} \).

**Case 3**: \( k > 1 \).
(a) If \( m \in (0, 1) \cup (k, \infty) \), then \( S_n \) is maximum for
\[
0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]
and is minimum for
\[
x_1 = \cdots = x_{j-1} = 0, \ x_{j+1} = \cdots = x_n,
\]
where \( j \in \{1, 2, \cdots, n\} \);
(b) If \( m \in (-\infty, 0) \cup (1, k) \), then \( S_n \) is minimum for
\[
0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]
and is maximum for
\[
x_1 = \cdots = x_{j-1} = 0, \ x_{j+1} = \cdots = x_n,
\]
where \( j \in \{1, 2, \cdots, n\} \).

The Equal Variable Theorem (EV-Theorem) for real variables has the following
statement (Vasile Cirtoaje, 2012).

**EV-Theorem** (for real variables). Let \( a_1, a_2, \cdots, a_n \ (n \geq 3) \) be fixed real numbers, let
\( x_1 \leq x_2 \leq \cdots \leq x_n \) be real variables such that
\[
x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,
\]
\[
x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,
\]
where \( k \) is an even positive integer, and let \( f \) be a differentiable function on \( \mathbb{R} \) such that the
associated function \( g : \mathbb{R} \to \mathbb{R} \) defined by
\[
g(x) = f'(k\sqrt{x})
\]
is strictly convex on \( \mathbb{R} \). Then, the sum
\[
S_n = f(x_1) + f(x_2) + \cdots + f(x_n)
\]
is minimum for \( x_2 = x_3 = \cdots = x_n \), and is maximum for \( x_1 = x_2 = \cdots = x_{n-1} \).

\[\square\]

21. ARITHMETIC COMPENSATION METHOD

The Arithmetic Compensation Theorem (AC-Theorem) has the following statement (Vasile Cirtoaje, 2002).

**AC-Theorem.** Let \( s > 0 \) and let \( F \) be a symmetric continuous function on the compact set in \( \mathbb{R}^n \)
\[
S = \{(x_1, x_2, \cdots, x_n) : x_1 + x_2 + \cdots + x_n = s, \ x_i \geq 0, \ i = 1, 2, \cdots, n\}.
\]
If
\[ F(x_1, x_2, x_3, \ldots, x_n) \geq \min \left\{ F \left( \frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n \right), F(0, x_1 + x_2, x_3, \ldots, x_n) \right\} \]
for all \((x_1, x_2, \ldots, x_n) \in S\), then \(F(x_1, x_2, x_3, \ldots, x_n)\) is minimal when
\[ x_1 = x_2 = \cdots = x_k = \frac{s}{k}, \quad x_{k+1} = \cdots = x_n = 0; \]
that is,
\[ F(x_1, x_2, x_3, \ldots, x_n) \geq \min_{1 \leq k \leq n} F \left( \frac{s}{k}, \ldots, \frac{s}{k}, 0, \ldots, 0 \right) \]
for all \((x_1, x_2, \ldots, x_n) \in S\).

Notice that if
\[ F(x_1, x_2, x_3, \ldots, x_n) < F \left( \frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n \right) \]
involves
\[ F(x_1, x_2, x_3, \ldots, x_n) \geq F(0, x_1 + x_2, x_3, \ldots, x_n), \]
then the hypothesis
\[ F(x_1, x_2, x_3, \ldots, x_n) \geq \]
\[ \geq \min \left\{ F \left( \frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n \right), F(0, x_1 + x_2, x_3, \ldots, x_n) \right\} \]
is satisfied.