The 1973 Putnam problem B-1 reads:

Let \( x_1, x_2, \ldots, x_{2n+1} \) be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of \( n \) numbers with equal sums. Prove that

\[
x_1 = x_2 = \cdots = x_{2n+1}.
\]

The problem has been known for a long time; at least, it was posed in the special case \( n = 6 \) (that is, for 13 numbers) for seventh- and eighth-graders at the 1949 Moscow Olympiad.

For brevity, call \( P \) the property described in the problem statement. It is “invariant under translation and dilatation”; that is, if \( x_1, \ldots, x_{2n+1} \) have the property \( P \), then so do \( a + x_1, a + x_2, \ldots, a + x_{2n+1} \) and \( bx_1, bx_2, \ldots, bx_{2n+1} \), where \( a \) and \( b \) are arbitrary real numbers. Thus we may assume \( x_1, x_2, \ldots, x_{2n+1} \) to be positive integers and go by induction, say on the greatest of them. If this is 1, then all of them equal 1 and we are done. Suppose now the statement holds for any collection of \( 2n + 1 \) positive integers not exceeding \( k \) (\( k \geq 2 \)), and take \( 2n + 1 \) positive integers \( x_1, x_2, \ldots, x_{2n+1} \) with the property \( P \) and not exceeding \( k + 1 \).

Let \( x_1 + x_2 + \cdots + x_{2n+1} = S \). Then \( P \) implies that all the numbers \( S - x_i \) are even, and so \( x_1, x_2, \ldots, x_{2n+1} \) are all of the same parity (the parity of \( S \)). If they are even, then \( x_1/2, x_2/2, \ldots, x_{2n+1}/2 \) are positive integers having the property \( P \). They do not exceed \((k + 1)/2\), which is less than or equal to \( k \) for \( k \geq 1 \). Thus \( x_1/2, x_2/2, \ldots, x_{2n+1}/2 \) are equal by the induction hypothesis, and hence so are \( x_1, x_2, \ldots, x_{2n+1} \). And if \( x_1, x_2, \ldots, x_{2n+1} \) are odd, the conclusion follows in a similar fashion, considering the numbers \((x_1 + 1)/2, (x_2 + 1)/2, \ldots, (x_{2n+1} + 1)/2\). ■
This proof depends heavily on the assumption that the given numbers are integers. Our aim here is to prove the following more general result:

Any $2n + 1$ real numbers having the property $P$ are equal.

The solution uses basic linear algebra. The property $P$ means that for each $i = 1, 2, \ldots, 2n + 1$ one can put 0 as a coefficient in front of $x_i$, then assign a coefficient equal to 1 or $-1$ to each of the remaining numbers $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{2n+1}$, so that $n$ of these coefficients are 1's, $n$ are $-1$'s and the algebraic sum obtained is 0. Formally speaking, for every $i = 1, 2, \ldots, 2n + 1$ there is a sequence $a_{i1}, a_{i2}, \ldots, a_{i, 2n+1}$ such that $a_{ii} = 0$, exactly $n$ of the remaining $a_{ij}$'s are 1's, exactly $n$ are $-1$'s and

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{i, 2n+1}x_{2n+1} = 0.$$  

In other words, $(x_1, x_2, \ldots, x_{2n+1})$ is a solution of a linear homogeneous system

$$\begin{align*}
 a_{11}x_1 + a_{12}x_2 + \cdots + a_{1, 2n+1}x_{2n+1} &= 0 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2, 2n+1}x_{2n+1} &= 0 \\
 &\vdots \\
 a_{2n+1, 1}x_1 + a_{2n+1, 2}x_2 + \cdots + a_{2n+1, 2n+1}x_{2n+1} &= 0
\end{align*}$$

whose matrix $A = (a_{ij})_{i,j=1}^{2n+1}$ has zeros on its main diagonal, ones or negative ones elsewhere, and each of its rows contains $n$ ones and $n$ negative ones.

The idea of this approach is to prove that $A$ has rank $2n$. For then the space of all solutions will have dimension $(2n + 1) - 2n = 1$ and it is enough to know one nonzero solution $(u_1, u_2, \ldots, u_{2n+1})$ in order to know them all: they must have the form $(\lambda u_1, \lambda u_2, \ldots, \lambda u_{2n+1})$ for some real $\lambda$. And since there is one obvious solution $(1, 1, \ldots, 1)$, any other will be of the form $(\lambda, \lambda, \ldots, \lambda)$!

There is no question that $A$ has determinant zero: adding all columns to the first one yields a column of straight zeros, because each row contains one zero and as many 1's as $-1$'s. And how to get a nonzero minor of order $2n$? The one in the upper left-hand corner can help. Moreover:

Any $2n \times 2n$ determinant $\Delta = |a_{ij}|_{i,j=1}^{2n}$ with even main diagonal entries and odd off-diagonal entries is different from 0.

Indeed, $\Delta$ is the algebraic sum of all products of the form

$$a_{i_1}a_{i_2} \cdots a_{i_2n}.$$
where \((i_1, i_2, \ldots, i_{2n})\) is an arbitrary permutation of \(\{1, 2, \ldots, 2n\}\). (We need not discuss how the signs + or − are assigned.) Since \(a_{ij}\) is even if and only if \(i = j\), a term of the above sum is odd exactly when the corresponding permutation \((i_1, i_2, \ldots, i_{2n})\) satisfies the conditions \(i_k \neq k\) for each \(k = 1, 2, \ldots, 2n\). Such permutations are called derangements; so it is enough to prove that the number of derangements of \(\{1, 2, \ldots, 2n\}\) is odd.

There are several well-known formulas for the number \(D(l)\) of derangements of the set \(\{1, 2, \ldots, l\}\), any of which could complete the proof:

\[
D(l) = l! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^l \frac{1}{l!} \right],
\]

\[
D(l) = lD(l - 1) + (-1)^l,
\]

\[
D(l) = (l - 1) \left[ D(l - 1) + D(l - 2) \right].
\]

But this solution can be done without referring to them. If a permutation \(\sigma\) of \(\{1, 2, \ldots, 2n\}\) is a derangement, then so is its inverse \(\sigma^{-1}\). Forget about \(\sigma\) and \(\sigma^{-1}\) if they are different; deleting them from the set of all derangements will not affect the parity of \(D(2n)\). Take a derangement \(\sigma\) with \(\sigma = \sigma^{-1}\). Note that \(\sigma\) is uniquely determined by some partition of \(\{1, 2, \ldots, 2n\}\) into \(n\) pairs \(\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\): it simply takes \(i_k\) to \(j_k\) and vice versa for all \(k = 1, 2, \ldots, n\). The pair \(\{i_1, j_1\}\) can be chosen in \(\binom{2n}{2}\) ways, the next pair \(\{i_2, j_2\}\) in \(\binom{2n - 2}{2}\) ways, and so on; there are \(\binom{2}{2} = 1\) possibilities for last pair \(\{i_n, j_n\}\). Thus there are

\[
\binom{2n}{2} \binom{2n - 2}{2} \cdots \binom{2}{2}
\]

ways to partition \(\{1, 2, \ldots, 2n\}\) in the fashion described above; but each of the actual partitions is counted \(n!\) times, because there are \(n!\) ways to permute the same pairs \(\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\). Hence the number of derangements in question is

\[
\frac{\binom{2n}{2} \binom{2n - 2}{2} \cdots \binom{2}{2}}{n!} = \frac{2n(2n - 1) \cdots 2 \cdot 1}{2 \cdot 4 \cdots (2n - 2) \cdot 2n} = 1 \cdot 3 \cdots (2n - 3)(2n - 1),
\]

and this is an odd integer. Our first solution is complete.

Here is a shorter alternate proof. It suffices to show that \(\Delta\) is odd, which will be unaffected if we reduce all entries modulo 2 and perform the computations mod 2. So we need to show that the \(2n \times 2n\) matrix \(M\)
with 0's on the diagonal and 1's elsewhere is nonsingular mod 2. Indeed, multiply $M$ by itself. By the definition of the product of matrices, one infers that each diagonal entry is odd, and each off-diagonal entry even. Hence $M^2$ is the identity matrix mod 2, so $M$ is nonsingular. □

These proofs show that the claim holds not only for reals but for complex numbers as well, and even for more general fields.

These are not the only possible approaches. There is, for example, a proof based on Diophantine approximations. Since it is technical and appeals to a certain nontrivial theorem, we are not going to discuss it here.