This equation has the general solution $x = 4t$, $y = 25 - 7t$, so that $z = 75 + 3t$, where t is an arbitrary integer. Chang himself gave several answers:

$$\begin{array}{lll} x = 4 & y = 18 & z = 78 \\ x = 8 & y = 11 & z = 81 \\ x = 12 & y = 4 & z = 84 \end{array}$$

A little further effort produces all solutions in the positive integers. For this, t must be chosen to satisfy simultaneously the inequalities

$$4t > 0 \qquad 25 - 7t > 0 \qquad 75 + 3t > 0$$

The last two of these are equivalent to the requirement $-25 < t < 3\frac{4}{7}$. Because t must have a positive value, we conclude that $t = 1, 2, 3$, leading to precisely the values Chang obtained.

PROBLEMS 2.5

1. Which of the following Diophantine equations cannot be solved?
(a) $6x + 51y = 22$.
(b) $33x + 14y = 115$.
(c) $14x + 35y = 93$.

2. Determine all solutions in the integers of the following Diophantine equations:
(a) $56x + 72y = 40$.
(b) $24x + 138y = 18$.
(c) $221x + 35y = 11$.

3. Determine all solutions in the positive integers of the following Diophantine equations:
(a) $18x + 5y = 48$.
(b) $54x + 21y = 906$.
(c) $123x + 360y = 99$.
(d) $158x - 57y = 7$.

4. If a and b are relatively prime positive integers, prove that the Diophantine equation $ax - by = c$ has infinitely many solutions in the positive integers.
[*Hint:* There exist integers x_0 and y_0 such that $ax_0 + by_0 = c$. For any integer t, which is larger than both $|x_0|/b$ and $|y_0|/a$, a positive solution of the given equation is $x = x_0 + bt$, $y = -(y_0 - at)$.]

5. (a) A man has \$4.55 in change composed entirely of dimes and quarters. What are the maximum and minimum number of coins that he can have? Is it possible for the number of dimes to equal the number of quarters?
(b) The neighborhood theater charges \$1.80 for adult admissions and \$.75 for children. On a particular evening the total receipts were \$90. Assuming that more adults than children were present, how many people attended?
(c) A certain number of sixes and nines is added to give a sum of 126; if the number of sixes and nines is interchanged, the new sum is 114. How many of each were there originally?

6. A farmer purchased 100 head of livestock for a total cost of \$4000. Prices were as follow: calves, \$120 each; lambs, \$50 each; piglets, \$25 each. If the farmer obtained at least one animal of each type, how many of each did he buy?

7. When Mr. Smith cashed a check at his bank, the teller mistook the number of cents for the number of dollars and vice versa. Unaware of this, Mr. Smith spent 68 cents and then

noticed to his surprise that he had twice the amount of the original check. Determine the smallest value for which the check could have been written.
[*Hint:* If x denotes the number of dollars and y the number of cents in the check, then $100y + x - 68 = 2(100x + y)$.]

8. Solve each of the puzzle-problems below:

(a) Alcuin of York, 775. One hundred bushels of grain are distributed among 100 persons in such a way that each man receives 3 bushels, each woman 2 bushels, and each child $\frac{1}{2}$ bushel. How many men, women, and children are there?

(b) Mahaviracarya, 850. There were 63 equal piles of plantain fruit put together and 7 single fruits. They were divided evenly among 23 travelers. What is the number of fruits in each pile?
[*Hint:* Consider the Diophantine equation $63x + 7 = 23y$.]

(c) Yen Kung, 1372. We have an unknown number of coins. If you make 77 strings of them, you are 50 coins short; but if you make 78 strings, it is exact. How many coins are there?
[*Hint:* If N is the number of coins, then $N = 77x + 27 = 78y$ for integers x and y.]

(d) Christoff Rudolff, 1526. Find the number of men, women, and children in a company of 20 persons if together they pay 20 coins, each man paying 3, each woman 2, and each child $\frac{1}{2}$.

(e) Euler, 1770. Divide 100 into two summands such that one is divisible by 7 and the other by 11.