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Problems with Solutions

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Problems

Problem 1. Determine all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that the equality

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z .)

Problem 2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D . Let E be a point on the arc \widehat{BDC} and F a point on the side BC such that

$$\angle BAF = \angle CAE < \frac{1}{2} \angle BAC.$$

Finally, let G be the midpoint of the segment IF . Prove that the lines DG and EI intersect on Γ .

Problem 3. Let \mathbb{N} be the set of positive integers. Determine all functions $g: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(g(m) + n)(m + g(n))$$

is a perfect square for all $m, n \in \mathbb{N}$.

Problem 4. Let P be a point inside the triangle ABC . The lines AP , BP and CP intersect the circumcircle Γ of triangle ABC again at the points K , L and M respectively. The tangent to Γ at C intersects the line AB at S . Suppose that $SC = SP$. Prove that $MK = ML$.

Problem 5. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:

Type 1: Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1} .

Type 2: Choose a nonempty box B_k with $1 \leq k \leq 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2} .

Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)

Problem 6. Let a_1, a_2, a_3, \dots be a sequence of positive real numbers. Suppose that for some positive integer s , we have

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\}$$

for all $n > s$. Prove that there exist positive integers ℓ and N , with $\ell \leq s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \geq N$.

Solutions

Problem 1. Determine all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that the equality

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor \quad (1)$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z .)

Answer. $f(x) = \text{const} = C$, where $C = 0$ or $1 \leq C < 2$.

Solution 1. First, setting $x = 0$ in (1) we get

$$f(0) = f(0) \lfloor f(y) \rfloor \quad (2)$$

for all $y \in \mathbb{R}$. Now, two cases are possible.

Case 1. Assume that $f(0) \neq 0$. Then from (2) we conclude that $\lfloor f(y) \rfloor = 1$ for all $y \in \mathbb{R}$. Therefore, equation (1) becomes $f(\lfloor x \rfloor y) = f(x)$, and substituting $y = 0$ we have $f(x) = f(0) = C \neq 0$. Finally, from $\lfloor f(y) \rfloor = 1 = \lfloor C \rfloor$ we obtain that $1 \leq C < 2$.

Case 2. Now we have $f(0) = 0$. Here we consider two subcases.

Subcase 2a. Suppose that there exists $0 < \alpha < 1$ such that $f(\alpha) \neq 0$. Then setting $x = \alpha$ in (1) we obtain $0 = f(0) = f(\alpha) \lfloor f(y) \rfloor$ for all $y \in \mathbb{R}$. Hence, $\lfloor f(y) \rfloor = 0$ for all $y \in \mathbb{R}$. Finally, substituting $x = 1$ in (1) provides $f(y) = 0$ for all $y \in \mathbb{R}$, thus contradicting the condition $f(\alpha) \neq 0$.

Subcase 2b. Conversely, we have $f(\alpha) = 0$ for all $0 \leq \alpha < 1$. Consider any real z ; there exists an integer N such that $\alpha = \frac{z}{N} \in [0, 1)$ (one may set $N = \lfloor z \rfloor + 1$ if $z \geq 0$ and $N = \lfloor z \rfloor - 1$ otherwise). Now, from (1) we get $f(z) = f(\lfloor N \rfloor \alpha) = f(N) \lfloor f(\alpha) \rfloor = 0$ for all $z \in \mathbb{R}$.

Finally, a straightforward check shows that all the obtained functions satisfy (1).

Solution 2. Assume that $\lfloor f(y) \rfloor = 0$ for some y ; then the substitution $x = 1$ provides $f(y) = f(1) \lfloor f(y) \rfloor = 0$. Hence, if $\lfloor f(y) \rfloor = 0$ for all y , then $f(y) = 0$ for all y . This function obviously satisfies the problem conditions.

So we are left to consider the case when $\lfloor f(a) \rfloor \neq 0$ for some a . Then we have

$$f(\lfloor x \rfloor a) = f(x) \lfloor f(a) \rfloor, \quad \text{or} \quad f(x) = \frac{f(\lfloor x \rfloor a)}{\lfloor f(a) \rfloor}. \quad (3)$$

This means that $f(x_1) = f(x_2)$ whenever $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$, hence $f(x) = f(\lfloor x \rfloor)$, and we may assume that a is an integer.

Now we have

$$f(a) = f(2a \cdot \frac{1}{2}) = f(2a) \lfloor f(\frac{1}{2}) \rfloor = f(2a) \lfloor f(0) \rfloor;$$

this implies $\lfloor f(0) \rfloor \neq 0$, so we may even assume that $a = 0$. Therefore equation (3) provides

$$f(x) = \frac{f(0)}{\lfloor f(0) \rfloor} = C \neq 0$$

for each x . Now, condition (1) becomes equivalent to the equation $C = C[C]$ which holds exactly when $[C] = 1$.

Problem 2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D . Let E be a point on the arc \widehat{BDC} and F a point on the side BC such that

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF . Prove that the lines DG and EI intersect on Γ .

Solution 1. Let X be the second point of intersection of line EI with Γ , and L be the foot of the bisector of angle BAC . Let G' and T be the points of intersection of segment DX with lines IF and AF , respectively. We are to prove that $G = G'$, or $IG' = G'F$. By the Menelaus theorem applied to triangle AIF and line DX , it means that we need the relation

$$1 = \frac{G'F}{IG'} = \frac{TF}{AT} \cdot \frac{AD}{ID}, \quad \text{or} \quad \frac{TF}{AT} = \frac{ID}{AD}.$$

Let the line AF intersect Γ at point $K \neq A$ (see Fig. 1); since $\angle BAK = \angle CAE$ we have $\widehat{BK} = \widehat{CE}$, hence $KE \parallel BC$. Notice that $\angle IAT = \angle DAK = \angle EAD = \angle EXD = \angle IXT$, so the points I, A, X, T are concyclic. Hence we have $\angle ITA = \angle IXA = \angle EXA = \angle EKA$, so $IT \parallel KE \parallel BC$. Therefore we obtain $\frac{TF}{AT} = \frac{IL}{AI}$.

Since CI is the bisector of $\angle ACL$, we get $\frac{IL}{AI} = \frac{CL}{AC}$. Furthermore, $\angle DCL = \angle DCB = \angle DAB = \angle CAD = \frac{1}{2}\angle BAC$, hence the triangles DCL and DAC are similar; therefore we get $\frac{CL}{AC} = \frac{DC}{AD}$. Finally, it is known that the midpoint D of arc BC is equidistant from points I, B, C , hence $\frac{DC}{AD} = \frac{ID}{AD}$.

Summarizing all these equalities, we get

$$\frac{TF}{AT} = \frac{IL}{AI} = \frac{CL}{AC} = \frac{DC}{AD} = \frac{ID}{AD},$$

as desired.

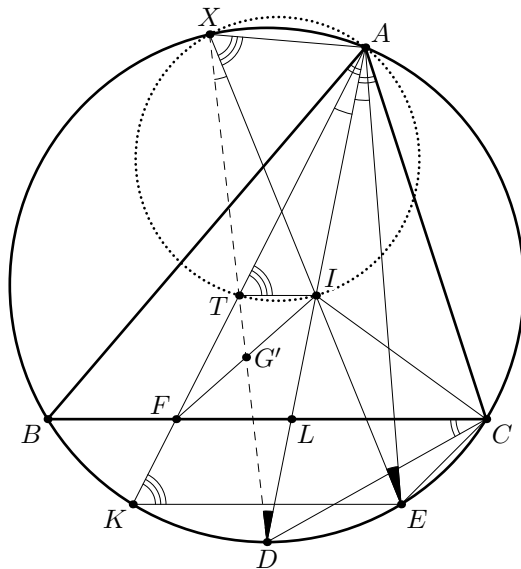


Fig. 1

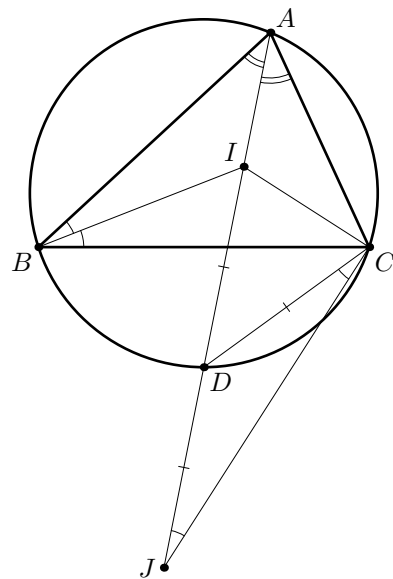


Fig. 2

Comment. The equality $\frac{AI}{IL} = \frac{AD}{DI}$ is known and can be obtained in many different ways. For instance, one can consider the inversion with center D and radius $DC = DI$. This inversion takes \widehat{BAC} to the segment BC , so point A goes to L . Hence $\frac{IL}{DI} = \frac{AI}{AD}$, which is the desired equality.

Solution 2. As in the previous solution, we introduce the points X, T and K and note that it suffice to prove the equality

$$\frac{TF}{AT} = \frac{DI}{AD} \iff \frac{TF + AT}{AT} = \frac{DI + AD}{AD} \iff \frac{AT}{AD} = \frac{AF}{DI + AD}.$$

Since $\angle FAD = \angle EAI$ and $\angle TDA = \angle XDA = \angle XEA = \angle IEA$, we get that the triangles ATD and AIE are similar, therefore $\frac{AT}{AD} = \frac{AI}{AE}$.

Next, we also use the relation $DB = DC = DI$. Let J be the point on the extension of segment AD over point D such that $DJ = DI = DC$ (see Fig. 2). Then $\angle DJC = \angle JCD = \frac{1}{2}(\pi - \angle JDC) = \frac{1}{2}\angle ADC = \frac{1}{2}\angle ABC = \angle ABI$. Moreover, $\angle BAI = \angle JAC$, hence triangles ABI and AJC are similar, so $\frac{AB}{AJ} = \frac{AI}{AC}$, or $AB \cdot AC = AJ \cdot AI = (DI + AD) \cdot AI$.

On the other hand, we get $\angle ABF = \angle ABC = \angle AEC$ and $\angle BAF = \angle CAE$, so triangles ABF and AEC are also similar, which implies $\frac{AF}{AC} = \frac{AB}{AE}$, or $AB \cdot AC = AF \cdot AE$.

Summarizing we get

$$(DI + AD) \cdot AI = AB \cdot AC = AF \cdot AE \implies \frac{AI}{AE} = \frac{AF}{AD + DI} \implies \frac{AT}{AD} = \frac{AF}{AD + DI},$$

as desired.

Comment. In fact, point J is an excenter of triangle ABC .

Problem 3. Let \mathbb{N} be the set of positive integers. Determine all functions $g: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(g(m) + n)(m + g(n))$$

is a perfect square for all $m, n \in \mathbb{N}$.

Answer. All functions of the form $g(n) = n + c$, where $c \in \mathbb{N} \cup \{0\}$.

Solution. First, it is clear that all functions of the form $g(n) = n + c$ with a constant nonnegative integer c satisfy the problem conditions since $(g(m) + n)(g(n) + m) = (n + m + c)^2$ is a square.

We are left to prove that there are no other functions. We start with the following

Lemma. Suppose that $p \mid g(k) - g(\ell)$ for some prime p and positive integers k, ℓ . Then $p \mid k - \ell$.

Proof. Suppose first that $p^2 \mid g(k) - g(\ell)$, so $g(\ell) = g(k) + p^2a$ for some integer a . Take some positive integer $D > \max\{g(k), g(\ell)\}$ which is not divisible by p and set $n = pD - g(k)$. Then the positive numbers $n + g(k) = pD$ and $n + g(\ell) = pD + (g(\ell) - g(k)) = p(D + pa)$ are both divisible by p but not by p^2 . Now, applying the problem conditions, we get that both the numbers $(g(k) + n)(g(n) + k)$ and $(g(\ell) + n)(g(n) + \ell)$ are squares divisible by p (and thus by p^2); this means that the multipliers $g(n) + k$ and $g(n) + \ell$ are also divisible by p , therefore $p \mid (g(n) + k) - (g(n) + \ell) = k - \ell$ as well.

On the other hand, if $g(k) - g(\ell)$ is divisible by p but not by p^2 , then choose the same number D and set $n = p^3D - g(k)$. Then the positive numbers $g(k) + n = p^3D$ and $g(\ell) + n = p^3D + (g(\ell) - g(k))$ are respectively divisible by p^3 (but not by p^4) and by p (but not by p^2). Hence in analogous way we obtain that the numbers $g(n) + k$ and $g(n) + \ell$ are divisible by p , therefore $p \mid (g(n) + k) - (g(n) + \ell) = k - \ell$.

□

We turn to the problem. First, suppose that $g(k) = g(\ell)$ for some $k, \ell \in \mathbb{N}$. Then by Lemma we have that $k - \ell$ is divisible by every prime number, so $k - \ell = 0$, or $k = \ell$. Therefore, the function g is injective.

Next, consider the numbers $g(k)$ and $g(k + 1)$. Since the number $(k + 1) - k = 1$ has no prime divisors, by Lemma the same holds for $g(k + 1) - g(k)$; thus $|g(k + 1) - g(k)| = 1$.

Now, let $g(2) - g(1) = q$, $|q| = 1$. Then we prove by induction that $g(n) = g(1) + q(n - 1)$. The base for $n = 1, 2$ holds by the definition of q . For the step, if $n > 1$ we have $g(n + 1) = g(n) \pm q = g(1) + q(n - 1) \pm q$. Since $g(n) \neq g(n - 2) = g(1) + q(n - 2)$, we get $g(n) = g(1) + qn$, as desired.

Finally, we have $g(n) = g(1) + q(n - 1)$. Then q cannot be -1 since otherwise for $n \geq g(1) + 1$ we have $g(n) \leq 0$ which is impossible. Hence $q = 1$ and $g(n) = (g(1) - 1) + n$ for each $n \in \mathbb{N}$, and $g(1) - 1 \geq 0$, as desired.

Problem 4. Let P be a point inside the triangle ABC . The lines AP , BP and CP intersect the circumcircle Γ of triangle ABC again at the points K , L and M respectively. The tangent to Γ at C intersects the line AB at S . Suppose that $SC = SP$. Prove that $MK = ML$.

Solution 1. We assume that $CA > CB$, so point S lies on the ray AB .

From the similar triangles $\triangle PKM \sim \triangle PCA$ and $\triangle PLM \sim \triangle PCB$ we get $\frac{PM}{KM} = \frac{PA}{CA}$ and $\frac{LM}{PM} = \frac{CB}{PB}$. Multiplying these two equalities, we get

$$\frac{LM}{KM} = \frac{CB}{CA} \cdot \frac{PA}{PB}.$$

Hence, the relation $MK = ML$ is equivalent to $\frac{CB}{CA} = \frac{PB}{PA}$.

Denote by E the foot of the bisector of angle B in triangle ABC . Recall that the locus of points X for which $\frac{XA}{XB} = \frac{CA}{CB}$ is the Apollonius circle Ω with the center Q on the line AB , and this circle passes through C and E . Hence, we have $MK = ML$ if and only if P lies on Ω , that is $QP = QC$.

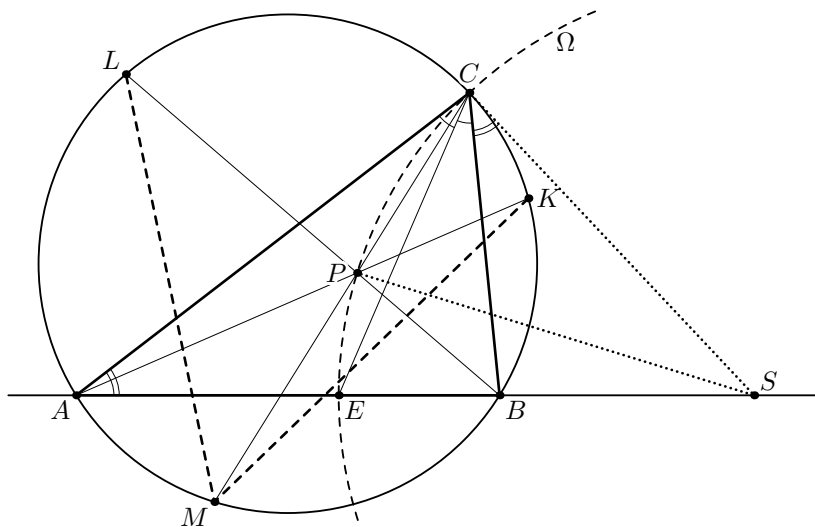


Fig. 1

Now we prove that $S = Q$, thus establishing the problem statement. We have $\angle CES = \angle CAE + \angle ACE = \angle BCS + \angle ECB = \angle ECS$, so $SC = SE$. Hence, the point S lies on AB as well as on the perpendicular bisector of CE and therefore coincides with Q .

Comment. In this solution we proved more general fact: $SC = SP$ if and only if $MK = ML$.

Solution 2. As in the previous solution, we assume that S lies on the ray AB .

Let P be an arbitrary point inside both the circumcircle ω of the triangle ABC and the angle ASC , the points K, L, M defined as in the problem.

Let E and F be the points of intersection of the line SP with ω , point E lying on the segment SP (see Fig. 2).

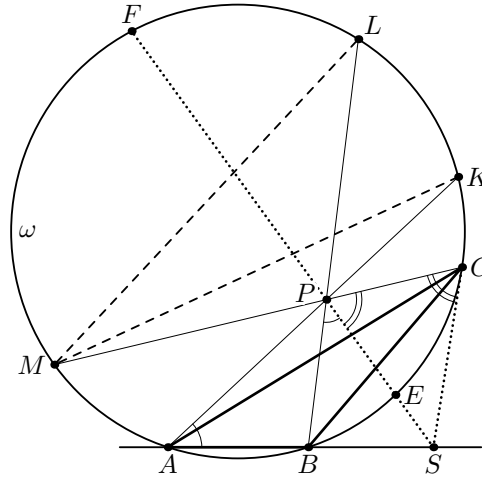


Fig. 2

We have $SP^2 = SC^2 = SA \cdot SB$, so $\frac{SP}{SB} = \frac{SA}{SP}$, and hence $\triangle PSA \sim \triangle BSP$. Then $\angle BPS = \angle SAP$. Since $2\angle BPS = \widehat{BE} + \widehat{LF}$ and $2\angle SAP = \widehat{BE} + \widehat{EK}$ we have

$$\widehat{LF} = \widehat{EK}. \quad (4)$$

On the other hand, from $\angle SPC = \angle SCP$ we have $\widehat{EC} + \widehat{MF} = \widehat{EC} + \widehat{EM}$, or

$$\widehat{MF} = \widehat{EM}. \quad (5)$$

From (4) and (5) we get $\widehat{MFL} = \widehat{MF} + \widehat{FL} = \widehat{ME} + \widehat{EK} = \widehat{MEK}$ and hence $MK = ML$. The claim is proved.

Problem 5. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:

Type 1: Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1} .

Type 2: Choose a nonempty box B_k with $1 \leq k \leq 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2} .

Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)

Answer. Yes. There exists such a sequence of moves.

Solution. Denote by $(a_1, a_2, \dots, a_n) \rightarrow (a'_1, a'_2, \dots, a'_n)$ the following: if some consecutive boxes contain a_1, \dots, a_n coins, then it is possible to perform several allowed moves such that the boxes contain a'_1, \dots, a'_n coins respectively, whereas the contents of the other boxes remain unchanged.

Let $A = 2010^{2010^{2010}}$, respectively. Our goal is to show that

$$(1, 1, 1, 1, 1, 1) \rightarrow (0, 0, 0, 0, 0, A).$$

First we prove two auxiliary observations.

Lemma 1. $(a, 0, 0) \rightarrow (0, 2^a, 0)$ for every $a \geq 1$.

Proof. We prove by induction that $(a, 0, 0) \rightarrow (a - k, 2^k, 0)$ for every $1 \leq k \leq a$. For $k = 1$, apply Type 1 to the first box:

$$(a, 0, 0) \rightarrow (a - 1, 2, 0) = (a - 1, 2^1, 0).$$

Now assume that $k < a$ and the statement holds for some $k < a$. Starting from $(a - k, 2^k, 0)$, apply Type 1 to the middle box 2^k times, until it becomes empty. Then apply Type 2 to the first box:

$$(a - k, 2^k, 0) \rightarrow (a - k, 2^k - 1, 2) \rightarrow \dots \rightarrow (a - k, 0, 2^{k+1}) \rightarrow (a - k - 1, 2^{k+1}, 0).$$

Hence,

$$(a, 0, 0) \rightarrow (a - k, 2^k, 0) \rightarrow (a - k - 1, 2^{k+1}, 0). \quad \square$$

Lemma 2. For every positive integer n , let $P_n = \underbrace{2^{2^{\cdot^{\cdot^2}}}}_n$ (e.g. $P_3 = 2^{2^2} = 16$). Then $(a, 0, 0, 0) \rightarrow (0, P_a, 0, 0)$ for every $a \geq 1$.

Proof. Similarly to Lemma 1, we prove that $(a, 0, 0, 0) \rightarrow (a - k, P_k, 0, 0)$ for every $1 \leq k \leq a$.

For $k = 1$, apply Type 1 to the first box:

$$(a, 0, 0, 0) \rightarrow (a - 1, 2, 0, 0) = (a - 1, P_1, 0, 0).$$

Now assume that the lemma holds for some $k < a$. Starting from $(a - k, P_k, 0, 0)$, apply Lemma 1, then apply Type 1 to the first box:

$$(a - k, P_k, 0, 0) \rightarrow (a - k, 0, 2^{P_k}, 0) = (a - k, 0, P_{k+1}, 0) \rightarrow (a - k - 1, P_{k+1}, 0, 0).$$

Therefore,

$$(a, 0, 0, 0) \rightarrow (a - k, P_k, 0, 0) \rightarrow (a - k - 1, P_{k+1}, 0, 0). \quad \square$$

Now we prove the statement of the problem.

First apply Type 1 to box 5, then apply Type 2 to boxes B_4 , B_3 , B_2 and B_1 in this order. Then apply Lemma 2 twice:

$$(1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 0, 3) \rightarrow (1, 1, 1, 0, 3, 0) \rightarrow (1, 1, 0, 3, 0, 0) \rightarrow (1, 0, 3, 0, 0, 0) \rightarrow \\ \rightarrow (0, 3, 0, 0, 0, 0) \rightarrow (0, 0, P_3, 0, 0, 0) = (0, 0, 16, 0, 0, 0) \rightarrow (0, 0, 0, P_{16}, 0, 0).$$

We already have more than A coins in box B_4 , since

$$A \leq 2010^{2010^{2010}} < (2^{11})^{2010^{2010}} = 2^{11 \cdot 2010^{2010}} < 2^{2010^{2011}} < 2^{(2^{11})^{2011}} = 2^{2^{11 \cdot 2011}} < 2^{2^{2^{15}}} < P_{16}.$$

To decrease the number of coins in box B_4 , apply Type 2 to this stack repeatedly until its size decreases to $A/4$. (In every step, we remove a coin from B_4 and exchange the empty boxes B_5 and B_6 .)

$$(0, 0, 0, P_{16}, 0, 0) \rightarrow (0, 0, 0, P_{16} - 1, 0, 0) \rightarrow (0, 0, 0, P_{16} - 2, 0, 0) \rightarrow \\ \rightarrow \dots \rightarrow (0, 0, 0, A/4, 0, 0).$$

Finally, apply Type 1 repeatedly to empty boxes B_4 and B_5 :

$$(0, 0, 0, A/4, 0, 0) \rightarrow \dots \rightarrow (0, 0, 0, 0, A/2, 0) \rightarrow \dots \rightarrow (0, 0, 0, 0, 0, A).$$

Comment. Starting with only 4 boxes, it is not hard to check manually that we can achieve at most 28 coins in the last position. However, around 5 and 6 boxes the maximal number of coins explodes. With 5 boxes it is possible to achieve more than $2^{2^{14}}$ coins. With 6 boxes the maximum is greater than $P_{P_{2^{14}}}$.

Problem 6. Let a_1, a_2, a_3, \dots be a sequence of positive real numbers. Suppose that for some positive integer s , we have

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \quad (6)$$

for all $n > s$. Prove that there exist positive integers ℓ and N , with $\ell \leq s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \geq N$.

Solution 1. First, from the problem conditions we have that each a_n ($n > s$) can be expressed as $a_n = a_{j_1} + a_{j_2}$ with $j_1, j_2 < n$, $j_1 + j_2 = n$. If, say, $j_1 > s$ then we can proceed in the same way with a_{j_1} , and so on. Finally, we represent a_n in a form

$$a_n = a_{i_1} + \dots + a_{i_k}, \quad (7)$$

$$1 \leq i_j \leq s, \quad i_1 + \dots + i_k = n. \quad (8)$$

Moreover, if a_{i_1} and a_{i_2} are the numbers in (7) obtained on the last step, then $i_1 + i_2 > s$. Hence we can adjust (8) as

$$1 \leq i_j \leq s, \quad i_1 + \dots + i_k = n, \quad i_1 + i_2 > s. \quad (9)$$

On the other hand, suppose that the indices i_1, \dots, i_k satisfy the conditions (9). Then, denoting $s_j = i_1 + \dots + i_j$, from (6) we have

$$a_n = a_{s_k} \geq a_{s_{k-1}} + a_{i_k} \geq a_{s_{k-2}} + a_{i_{k-1}} + a_{i_k} \geq \dots \geq a_{i_1} + \dots + a_{i_k}.$$

Summarizing these observations we get the following

Claim. For every $n > s$, we have

$$a_n = \max\{a_{i_1} + \cdots + a_{i_k} : \text{the collection } (i_1, \dots, i_k) \text{ satisfies (9)}\}. \quad \square$$

Now we denote

$$m = \max_{1 \leq i \leq s} \frac{a_i}{i}$$

and fix some index $\ell \leq s$ such that $m = \frac{a_\ell}{\ell}$.

Consider some $n \geq s^2\ell + 2s$ and choose an expansion of a_n in the form (7), (9). Then we have $n = i_1 + \cdots + i_k \leq sk$, so $k \geq n/s \geq s\ell + 2$. Suppose that none of the numbers i_3, \dots, i_k equals ℓ . Then by the pigeonhole principle there is an index $1 \leq j \leq s$ which appears among i_3, \dots, i_k at least ℓ times, and surely $j \neq \ell$. Let us delete these ℓ occurrences of j from (i_1, \dots, i_k) , and add j occurrences of ℓ instead, obtaining a sequence $(i_1, i_2, i'_3, \dots, i'_{k'})$ also satisfying (9). By Claim, we have

$$a_{i_1} + \cdots + a_{i_k} = a_n \geq a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}},$$

or, after removing the coinciding terms, $\ell a_j \geq j a_\ell$, so $\frac{a_\ell}{\ell} \leq \frac{a_j}{j}$. By the definition of ℓ , this means that $\ell a_j = j a_\ell$, hence

$$a_n = a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}}.$$

Thus, for every $n \geq s^2\ell + 2s$ we have found a representation of the form (7), (9) with $i_j = \ell$ for some $j \geq 3$. Rearranging the indices we may assume that $i_k = \ell$.

Finally, observe that in this representation, the indices (i_1, \dots, i_{k-1}) satisfy the conditions (9) with n replaced by $n - \ell$. Thus, from the Claim we get

$$a_{n-\ell} + a_\ell \geq (a_{i_1} + \cdots + a_{i_{k-1}}) + a_\ell = a_n,$$

which by (6) implies

$$a_n = a_{n-\ell} + a_\ell \quad \text{for each } n \geq s^2\ell + 2s,$$

as desired.

Solution 2. As in the previous solution, we involve the expansion (7), (8), and we fix some index $1 \leq \ell \leq s$ such that

$$\frac{a_\ell}{\ell} = m = \max_{1 \leq i \leq s} \frac{a_i}{i}.$$

Now, we introduce the sequence (b_n) as $b_n = a_n - mn$; then $b_\ell = 0$.

We prove by induction on n that $b_n \leq 0$, and (b_n) satisfies the same recurrence relation as (a_n) . The base cases $n \leq s$ follow from the definition of m . Now, for $n > s$ from the induction hypothesis we have

$$b_n = \max_{1 \leq k \leq n-1} (a_k + a_{n-k}) - nm = \max_{1 \leq k \leq n-1} (b_k + b_{n-k} + nm) - nm = \max_{1 \leq k \leq n-1} (b_k + b_{n-k}) \leq 0,$$

as required.

Now, if $b_k = 0$ for all $1 \leq k \leq s$, then $b_n = 0$ for all n , hence $a_n = mn$, and the statement is trivial. Otherwise, define

$$M = \max_{1 \leq i \leq s} |b_i|, \quad \varepsilon = \min\{|b_i| : 1 \leq i \leq s, b_i < 0\}.$$

Then for $n > s$ we obtain

$$b_n = \max_{1 \leq k \leq n-1} (b_k + b_{n-k}) \geq b_\ell + b_{n-\ell} = b_{n-\ell},$$

so

$$0 \geq b_n \geq b_{n-\ell} \geq b_{n-2\ell} \geq \cdots \geq -M.$$

Thus, in view of the expansion (7), (8) applied to the sequence (b_n) , we get that each b_n is contained in a set

$$T = \{b_{i_1} + b_{i_2} + \cdots + b_{i_k} : i_1, \dots, i_k \leq s\} \cap [-M, 0]$$

We claim that this set is finite. Actually, for any $x \in T$, let $x = b_{i_1} + \cdots + b_{i_k}$ ($i_1, \dots, i_k \leq s$). Then among b_{i_j} 's there are at most $\frac{M}{\varepsilon}$ nonzero terms (otherwise $x < \frac{M}{\varepsilon} \cdot (-\varepsilon) < -M$). Thus x can be expressed in the same way with $k \leq \frac{M}{\varepsilon}$, and there is only a finite number of such sums.

Finally, for every $t = 1, 2, \dots, \ell$ we get that the sequence

$$b_{s+t}, b_{s+t+\ell}, b_{s+t+2\ell}, \dots$$

is non-decreasing and attains the finite number of values; therefore it is constant from some index. Thus, the sequence (b_n) is periodic with period ℓ from some index N , which means that

$$b_n = b_{n-\ell} = b_{n-\ell} + b_\ell \quad \text{for all } n > N + \ell,$$

and hence

$$a_n = b_n + nm = (b_{n-\ell} + (n - \ell)m) + (b_\ell + \ell m) = a_{n-\ell} + a_\ell \quad \text{for all } n > N + \ell,$$

as desired.