PROBLEMS AND SOLUTIONS

Edited by Gerald A. Edgar, Doug Hensley, Douglas B. West

Proposed problems and solutions should be sent in duplicate to the MONTHLY problems address on the inside front cover. Submitted solutions should arrive at that address before June 30, 2008. Additional information, such as generalizations and references, is welcome. The problem number and the solver’s name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11341. Proposed by Cezar Lupu, University of Bucharest, Bucharest, Romania (student), and Tudorel Lupu, Decebal High School, Constanza, Romania. Consider an acute triangle with side-lengths \( a \), \( b \), and \( c \), with inradius \( r \) and semiperimeter \( p \). Show that

\[
(1 - \cos A)(1 - \cos B)(1 - \cos C) \geq \cos A \cos B \cos C \left( 2 - \frac{3\sqrt{3}r}{p} \right).
\]

11342. Proposed by Luis H. Gallardo, University of Brest, Brest, France. Let \( p \) be a prime and let \( \mathbb{F}_q \) be a finite field of characteristic \( p \), where \( q \) is a power of \( p \). Let \( n \) be a divisor of \( q - 1 \). With the natural mapping of \( \mathbb{Z} \) onto \( \mathbb{F}_p \) and embedding of \( \mathbb{F}_p \) in \( \mathbb{F}_q \), show that \((-1)^{(\alpha+2)(n-1)/2}n^\alpha\) is a square in \( \mathbb{F}_q \).

11343. Proposed by David Beckwith, Sag Harbor, NY. Show that when \( n \) is a positive integer,

\[
\sum_{k \geq 0} \binom{n}{k} \binom{2k}{k} = \sum_{k \geq 0} \binom{n}{2k} \binom{2k}{k} 3^{n-2k}.
\]

11344. Proposed by Albert Stadler, Meilen, Switzerland. Let \( \mu \) be the Möbius \( \mu \) function of number theory. Show that if \( n \) is a positive integer and \( n > 1 \), then

\[
\sum_{j=1}^{n} \mu(j) = -\sum_{j=1}^{\left\lfloor (n-1)/2 \right\rfloor} j \sum_{k=\left\lfloor (n+1)/(2j+3) \right\rfloor}^{\left\lfloor n/(2j+1) \right\rfloor} \mu(k).
\]

11345. Proposed by Roger Cuculiète, France. Find all nondecreasing functions \( f \) from \( \mathbb{R} \) to \( \mathbb{R} \) such that \( f(x + f(y)) = f(f(x)) + f(y) \) for all real \( x \) and \( y \).

11346. Proposed by Christopher Hillar, Texas A&M University, College Station, TX, and Lionel Levine, University of California, Berkeley, CA. Let \( n \) be an integer greater
than 1 and let $S = \{2, \ldots, n\}$. For each nonempty subset $A$ of $S$, let $\pi(A) = \prod_{j \in A} j$.
Prove that when $k$ is a positive integer and $k < n$,

$$\prod_{i=k}^{n} \operatorname{lcm}\{1, \ldots, [n/i]\} = \gcd((\pi(A) : |A| = n - k)).$$

(In particular, setting $k = 1$ yields $\prod_{i=1}^{n} \operatorname{lcm}\{1, \ldots, [n/i]\} = n!$)

**11347. Proposed by Mihály Bencze, Brasov, Romania.** Let $A = (x + y)/2$, $G = \sqrt{xy}$, and

$$I = \frac{1}{e} \left( \frac{x^y}{y^x} \right)^{1/(x-y)}.$$

Determine all ordered 4-tuples $(\alpha, \beta, \gamma, \delta)$ of positive numbers with $\alpha > \beta$ and $\gamma > \delta$ such that for all distinct positive $x$ and $y$,

$$I > \frac{\alpha A + \beta G}{\alpha + \beta} > (A^\gamma G^\delta)^{1/(\gamma + \delta)} > \sqrt{AG}.$$

**SOLUTIONS**

**Orthogonal Sudoku Squares**

**11214 [2006, 268]. Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, CA.** A Sudoku solution is a $9 \times 9$ square array with integer entries such that each of the nine possible entries occurs exactly once in each row, once in each column, and once in each of the nine $3 \times 3$ subsquares that together tile the main array. Is it possible for two Sudoku solutions to form a pair of orthogonal Latin squares? (That is, can the 81 pairs of corresponding cells contain all 81 possible pairs of entries?)

*Solution by Kyle Calderhead, Illinois College, Jacksonville, IL.* Yes. In fact, there is a family of six pairwise orthogonal Latin Sudoku squares:

It comes from the usual construction of complete families of orthogonal Latin squares, limited to the ones that also meet the Sudoku restrictions. The standard construction permutes the rows of the addition table for the finite field $\mathbb{F}_9$, with the permutations determined by the (nonzero) rows of the multiplication table. If we represent $\mathbb{F}_9$ as the set of congruence classes of polynomials over $\mathbb{F}_3$ modulo a quadratic polynomial such as $x^2 + 1$ that is irreducible over $\mathbb{F}_3$, then the rows of the multiplication table that result in Latin Sudoku squares are precisely those that correspond to nonconstant polynomials. This produces the six Latin Sudoku squares shown below. More generally, for any prime $p$, the same technique will produce a family of $p(p - 1)$ mutually orthogonal $p^2 \times p^2$ Latin Sudoku squares.

$$\begin{array}{ccc|ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \\
7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 \\
6 & 4 & 5 & 9 & 7 & 8 & 3 & 1 & 2 \\
9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \\
\hline
2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 \\
5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1 \\
8 & 9 & 7 & 2 & 3 & 1 & 5 & 6 & 4 \\
\end{array}$$

$$\begin{array}{ccc|ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1 \\
9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \\
\hline
6 & 4 & 5 & 9 & 7 & 8 & 3 & 1 & 2 \\
7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 \\
\hline
8 & 9 & 7 & 2 & 3 & 1 & 5 & 6 & 4 \\
3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 \\
4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \\
\end{array}$$

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Editorial comment. Lyle Ramshaw provided a proof that 6 is the maximum number of pairwise orthogonal Latin squares of order 9. These results and others are dealt with in “Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes” by R. A. Bailey, Peter J. Cameron, and Robert Connelly, to appear shortly in this MONTHLY.

Also solved by R. Bagby, J. Brawner, R. Chapman (U. K.), K. Dale (Norway), D. Degiorgi (Switzerland), G. T. Fala, R. T. Guy, D. E. Knuth, O. P. Lossers (Netherlands), M. D. Meyerson & T. S. Michael, R. M. Pedersen, L. Ramshaw, T. Q. Sibley, J. H. Steelman, R. Stong, H. Stubbs, R. Tauraso (Italy), L. Wenstrom, the ABC Student Problem Solving Group, the BSI Problems Group (Germany), Szeged Problem Solving Group “Fejértalálta”(Hungary), the GCHQ Problem Solving Group (U. K.), the Northwestern University Math Problem Solving Group, the VMI Problem Solving Group, and the proposer.

Maximum Velocity on a Car Trip

11215 [2006, 366]. Proposed by Shmuel Rosset, Tel Aviv University, Tel Aviv, Israel. A car moves along the real line from \( x = 0 \) at \( t = 0 \) to \( x = 1 \) at \( t = 1 \), with differentiable position function \( x(t) \) and differentiable velocity function \( v(t) = x'(t) \). The car begins and ends the trip at a standstill; that is, \( v = 0 \) at both the beginning and the end of the trip. Let \( L \) be the maximum velocity attained during the trip. Prove that at some time between the beginning and the end of the trip, \( |v'| > L^2/(L - 1) \).

Solution by José Heber Nieto, Universidad del Zulia, Moracaibo, Venezuela. Let \( M = L^2/(L - 1) \). If \( |v'| \leq M \) for all \( t \in [0, 1] \), then \( v(t) = \int_0^t v'(u) \, du \leq Mt \) and \( v(t) = -\int_1^t v'(u) \, du \leq M(1 - t) \) for all \( t \in [0, 1] \). Since \( L = v(t_0) \) for some \( t_0 \in [0, 1] \), we have \( L \leq Mt_0 \) and \( L \leq M(1 - t_0) \). Thus \( 1 - \frac{1}{L} \leq t_0 \) and \( t_0 \leq \frac{1}{L} \), and therefore \( L \leq 2 \).

Now consider the function \( f \) defined by

\[
  f(t) = \begin{cases} 
    Mt, & \text{if } 0 \leq t < 1 - \frac{1}{L}, \\
    L, & \text{if } 1 - \frac{1}{L} \leq t < \frac{1}{L}, \\
    M(1 - t), & \text{if } \frac{1}{L} \leq t \leq 1.
  \end{cases}
\]

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Observe that
\[ \int_0^1 f(t) \, dt = \frac{L}{2} \left( 1 - \frac{1}{L} \right) + L \left( \frac{2}{L} - 1 \right) + \frac{L}{2} \left( 1 - \frac{1}{L} \right) = 1. \]

We have \( v(t) \leq f(t) \) for \( t \in [0, 1] \). Equality cannot hold everywhere, since \( f \) is not differentiable at \( 1/L \). Hence \( v(t_1) < f(t_1) \) at some \( t_1 \in [0, 1] \) and, by continuity, \( v(t) < f(t) \) in some neighborhood of \( t_1 \). Our assumption that \( |v'| \leq M \) on \( [0, 1] \) thus leads to this contradiction:
\[ 1 = x(1) - x(0) = \int_0^1 v(t) \, dt < \int_0^1 f(t) \, dt = 1. \]


**Goes Back to Boole**

11234 [2006, 568]. Proposed by Jim Brennan and Richard Ehrenborg, University of Kentucky, Lexington, KY. Let \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_{n-1} \) be real numbers, with \( a_1 < b_1 < a_2 < \ldots < a_{n-1} < b_{n-1} < a_n \). Let \( h \) be an integrable function from \( \mathbb{R} \) to \( \mathbb{R} \). Show that
\[ \int_{-\infty}^{\infty} h \left( \frac{(x-a_1) \cdots (x-a_n)}{(x-b_1) \cdots (x-b_{n-1})} \right) \, dx = \int_{-\infty}^{\infty} h(x) \, dx. \]

**Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.** Let
\[ f(x) = \frac{(x-a_1) \cdots (x-a_n)}{(x-b_1) \cdots (x-b_{n-1})}. \]

On each of the open intervals
\[ (-\infty, b_1), (b_1, b_2), \ldots, (b_{n-1}, \infty) \quad (1) \]
the function \( f \) is continuous and has limits \(-\infty\) and \( \infty \) at the left and right endpoints, respectively. Thus the range of \( f \) on each of these intervals is \((-\infty, \infty)\). Hence, for every real number \( y \), the equation \( y = f(x) \) has a solution in each of these \( n \) intervals. In fact, there is exactly one solution in each interval, since on these intervals, \( f(x) - y = 0 \) if and only if
\[ (x-a_1) \cdots (x-a_n) - y (x-b_1) \cdots (x-b_{n-1}) = 0, \quad (2) \]
which is a polynomial equation in \( x \) of degree \( n \). Therefore, on each of these \( n \) intervals, the function \( f \) is invertible. We denote the inverse of \( f \) on the \( i \)th interval by \( g_i \). In particular, for any \( y \in \mathbb{R} \), the numbers \( g_1(y), \ldots, g_n(y) \) are the solutions of the polynomial equation (2). Consequently, the sum of these solutions is the negative of the coefficient of \( x^{n-1} \) in (2); that is,
\[ \sum_{i=1}^{m} g_i(y) = y + \sum_{i=1}^{n} a_i, \quad (3) \]
We transform the integral \( \int_{-\infty}^{\infty} h(f(x)) \, dx \) by making the substitution \( y = f(x) \) in each of the \( n \) intervals (1); that is, in the \( i \)th interval, we have \( x = g_i(y) \) and hence \( dx = g_i'(y) \, dy \). Therefore

\[
\int_{-\infty}^{\infty} h(f(x)) \, dx = \int_{-\infty}^{b_1} h(f(x)) \, dx + \int_{b_1}^{b_2} h(f(x)) \, dx + \cdots + \int_{b_{n-1}}^{\infty} h(f(x)) \, dx
\]

\[
= \sum_{i=1}^{n} \int_{-\infty}^{b_i} h(y) g_i'(y) \, dy = \int_{-\infty}^{\infty} h(y) \sum_{i=1}^{n} g_i'(y) \, dy = \int_{-\infty}^{\infty} h(y) \, dy.
\]

The last step follows from equation (3).

**Editorial comment.** Six solvers noted that this result is a consequence of the following theorem (*): If \( h(x) \) is integrable and \( g(x) = x - \sum_{i=1}^{m} c_i/(x - b_i) \) where all the \( c_i \) are positive, then \( \int_{-\infty}^{\infty} h(g(x)) \, dx = \int_{-\infty}^{\infty} h(x) \, dx \). This reduction is obtained by writing \( (x-a_1)\cdots(x-a_n)/(x-b_0) \cdots(x-b_{n-1}) \) in its partial fraction form \( g(x) = x + C - \sum_{i=1}^{m} c_i/(x - b_i) \), using the given conditions to show that all the \( c_i \) are positive, then appealing to theorem (*) and to the translation invariance of Lebesgue measure. References provided for theorem (*) were: (1) G. Boole, “On a general theorem of definite integration,” *Cambridge & Dublin Mathematical Journal*, 1849; G. Boole, “On the comparison of transcendent s . . . ” *Philos. Trans. Roy. Soc., London* 147 (1857) 745–803 (the proof perhaps not meeting modern standards of rigor); (2) M. L. Glasser, “A remarkable property of definite integrals,” *Math. Comput.* 40 (1983) 561–563; (3) G. Pólya & G. Szegő, *Problems and Theorems in Analysis*, vol. 1 (Springer, New York, 1972), Problem II-12.


**Coloring the Plane**

**11236 [2006, 655]. Proposed by Jim Owings, Riderwood, MD.** Given a positive integer \( n \) and a positive real number \( x \), consider the proposition \( P(x, n) \): If we color each point in the Euclidean plane with one of \( n \) colors, then there exist two points of the same color that are either distance 1 or distance \( x \) apart.

(a) Prove that \( P((1 + \sqrt{5})/2, 4) \), \( P(\sqrt{3}, 4) \), and \( P(\sqrt{2}, 4) \) all hold.

(b) Does \( P(t, 4) \) hold for any \( t > 1 \) other than those specified in part (a)?

(c) Does there exist \( t > 1 \) such that \( P(t, 5) \) holds?

**Solution to (a) and (b) by Marian Tetiva, Bârlad, Romania.** In a regular pentagon with side 1, any two of the vertices have distance 1 or \((1 + \sqrt{5})/2\). If the plane is colored in four colors, then two of these five points must have the same color. This proves \( P((1 + \sqrt{5})/2, 4) \).

Next consider \( P(\sqrt{3}, 4) \). Assume there is a 4-coloring of the plane such that any two points separated by a distance of 1 or \( \sqrt{3} \) have different colors. Let \( A \) and \( B \) be any two points with distance 2. Choose \( C \) so that \( ABC \) is an equilateral triangle, and let \( D, E, F \) be the midpoints of the segments \( BC, CA, AB \), respectively. The distance between any two of the resulting four points \( A, D, E, F \) is 1 or \( \sqrt{3} \), so they have
different colors. Similarly $B, D, E, F$ have different colors. Thus $A$ and $B$ have the same color. Since $B$ was an arbitrary point on the circle with center $A$ and radius 2, that circle is monochromatic. This gives a contradiction, because there are two points on that circle with distance 1.

Next we prove $P((1 + \sqrt{3})/\sqrt{2}, 4)$; since this is not listed in (a), it will be an example for (b). It is equivalent to prove: In every 4-coloring of the Euclidean plane there are two points having the same color that are distance $1 + \sqrt{3}$ or distance $\sqrt{2}$ apart. Let $A$ and $B$ be any two points with distance 2. Let $AB$ be the hypotenuse of isosceles right triangle $ABC$ so that $AC = BC = \sqrt{2}$. On the circle centered at $A$ with radius $\sqrt{2}$, take $D$ and $E$ to be the two points such that $\angle DAB = \angle EAB = 105$ degrees. Now $DBE$ is an equilateral triangle with sides of length $1 + \sqrt{3}$, and $ACD$ is an equilateral triangle with sides of length $\sqrt{2}$.

In this configuration $BC = AC = AD = AE = CD = \sqrt{2}$, while $BD = DE = EB = EC = 1 + \sqrt{3}$. Assuming a 4-coloring of the plane contrary to the claim, the four points $A, C, D,$ and $E$ have different colors, as do the four points $B, C, D,$ and $E$. Therefore $A$ and $B$ have the same color. Since $A$ and $B$ were arbitrary points with distance 2, the entire circle centered at $A$ with radius 2 has just one color. This gives a contradiction, because there are two points on that circle separated by distance $\sqrt{2}$.

Next consider $P(\sqrt{2}, 4)$. Assume a 4-coloring of the plane inconsistent with $P(\sqrt{2}, 4)$, that is, one in which any two points with distance 1 or $\sqrt{2}$ have different colors. By $P((1 + \sqrt{3})/\sqrt{2}, 4)$, just proved, since any two points with distance $\sqrt{2}$ have different colors, there must exist two points $X$ and $T$ with distance $1 + \sqrt{3}$ having the same color; call it color 1. Let $XYZTUV$ be a convex hexagon such that points $Y$ and $V$ are symmetric with respect to line $XT$, points $Z$ and $U$ are symmetric with respect to line $XT$, and $\triangle XYZ$ and $\triangle TZU$ are equilateral triangles with side 1. Now $Y, Z, U,$ and $V$ all have colors different from color 1, so some two of them have the same color. This gives a contradiction, because $YZUV$ forms a square with side 1 and diagonal $\sqrt{2}$.

**Solution to (c)** by Matthew Huddleston, Washington State University, Pullman, WA.

We prove $P((1 + \sqrt{5})/2, 5)$. Suppose there is a 5-coloring of the plane so that no two points with distance 1 or $(1 + \sqrt{5})/2$ have the same color. Let $S$ be a set of five points forming the vertices of a regular pentagon with side 1. Let $Q$ be the set of ordered quintuples chosen from $S$, so that $Q$ has $5^5$ elements. Color $Q$ by assigning to each of its 5-tuples the color, in the original coloring, of the sum of its five entries.

For any 4-tuple of points in $S$, note that adding the sum of its entries to each of the five points of $S$ gives a regular pentagon with side 1, so these five points have different colors. Therefore, each of the five colors is assigned to $5^4$ of the $5^5$ elements of $Q$. On the other hand, permuting an ordered 5-tuple in $Q$ cannot change its color. The number
of permutations of a given quintuple is a multinomial coefficient of the form
\[
\binom{5}{a_1, a_2, a_3, a_4, a_5} = \frac{5!}{a_1! a_2! a_3! a_4! a_5!},
\]
where \(a_j\) is the number of occurrences in the quintuple of the \(j\)th element of \(S\). This
multinomial coefficient is a multiple of 5 except for the cases in which one of the \(a_j\) is
5 and the rest are 0. In order for the sizes of all the color classes in \(Q\) to be multiples
of 5, these 5 exceptional cases must all be assigned the same color. Equivalently, the
points of 5S all have the same color. This shows that in any regular pentagon with side
5 all vertices have the same color, so that any two vertices with distance 5 have the
same color. An isosceles triangle with sides of length 5, 5, 1 thus has vertices of the
same color, and that is a contradiction.

**Editorial comment.** All solvers for Part (b) found the same example: \((1 + \sqrt{3})/\sqrt{2} =
(\sqrt{6} + \sqrt{2})/2 = \sqrt{2} + \sqrt{3} = (1/2) \csc(\pi/12) = 2 \sin(5\pi/12) = 2 \cos(\pi/12)\). Is
there another \(t > 1\) such that \(P(t, 4)\) holds?

Parts (a) and (b) also solved by W. C. Calhoun, R. Stong, the GCHQ Problem Solving Group (U. K.),
and Microsoft Research Problems Group. Part (a) also solved by O. P. Lossers (Netherlands) and the proposer.

**Ex-To-In-Radius Ratio**

11240 [2006, 655]. **Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.** Let \(a, b,\) and \(c\) be the lengths of the sides of a triangle, and let \(R\) and \(r\) be
the circumradius and inradius of that triangle, respectively. Show that
\[
\frac{R}{2r} \geq \exp\left(\frac{(a - b)^2}{2c^2} + \frac{(b - c)^2}{2a^2} + \frac{(c - a)^2}{2b^2}\right).
\]

**Solution by A. K. Shafie, IASBS, Zanjan, Iran.** Write \(A, B, C\) for the angles opposite
sides \(a, b, c\), respectively, and \(s\) for the semiperimeter. We have
\[
r = \frac{s - a}{\tan \frac{A}{2}} = a \sec \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2},
\]
so\(\sin(A/2) \sin(B/2) \sin(C/2) = r/(4R)\). Now \(e^x \leq 1/(1 - x)\) for all \(x \in [0, 1)\).
Since \(0 \leq (a - b)^2/c^2 < 1\) and
\[
1 - \frac{(a - b)^2}{c^2} = \frac{a^2 + b^2 - 2ab \cos C - a^2 - b^2 + 2ab}{c^2} = \frac{4ab \sin^2(C/2)}{c^2},
\]
we have
\[
\exp\left(\frac{(a - b)^2}{c^2} + \frac{(a - c)^2}{b^2} + \frac{(b - c)^2}{a^2}\right) \leq \frac{c^2}{4ab \sin^2(C/2)} \frac{b^2}{4ac \sin^2(B/2)} \frac{a^2}{4bc \sin^2(A/2)} = \frac{R^2}{4r^2}.
\]
This is the square of the required inequality.

Also solved by A. Alt, S. Amghibech (Canada), M. R. Avidon, O. Bagdasar (Romania), E. Braune (Austria), P. De (Ireland), Y. Dumont (France), O. Faynshteyn (Germany), S. Hitotumatu (Japan), A. Ilić & M. Novaković (Serbia), K. W. Lau (China), O. P. Lossers (Netherlands), D. Lovit, M. Mabuchi (Japan), J. Minkus, J. Rooin & F. Ghanimat (Iran), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), M. Vowe (Switzerland), J. Zacharias, B. Zhao, BSI Problems Group (Germany), Szeged Problem Solving Group “Fejéntaláltuka” (Hungary), Microsoft Research Problems Group, and the proposer.
Iterational Rate of Convergence

11244 [2006, 759]. Proposed by Jolene Harris and Bogdan Suceavă, California State University, Fullerton, CA. Let \( f \) be a differentiable function from the positive reals to the positive reals with the property that \( f(x) < x \) for all \( x \). Suppose that \( x_1 > 0 \), and for \( n > 1 \) let \( x_n = f(x_{n-1}) \). Suppose further that \( \lim_{n \to \infty} x_n = 0 \) and that there exist positive numbers \( a \) and \( k \) such that

\[
\lim_{x \to 0} \frac{x^a - (f(x))^a}{x^a (f(x))^a} = \frac{1}{k^a}.
\]

(a) Prove that \( \lim_{n \to \infty} n^{1/a} x_n = k \).
(b) Suppose that \( 0 < x_1 < 1 \), and specialize to the case where \( f \) is given by \( f(x) = \sin x \) if \( x < \pi/2 \) and \( f(x) = 1 \) if \( x \geq \pi/2 \). Show that \( \lim_{n \to \infty} x_n \sqrt{n} = \sqrt{3} \).
(c) Finally, suppose instead that \( 0 < x_1 < 1 \) and \( f(x) = 1 - e^{-x} \). Show that, in this case, \( \lim_{n \to \infty} n x_n = 2 \).

Solution by Knut Dale, Telemark University College, Bø, Norway. (a) We are given \( f(x)^{-a} - x^{-a} = k^{-a} + \epsilon(x) \), where \( \epsilon(x) \to 0 \) as \( x \to 0 \). So

\[
\frac{1}{x_n^a} - 1 \leq \frac{1}{x_{i+1}^a} - \frac{1}{x_i^a} = \frac{n - 1}{k^a} + \epsilon(x_1) + \cdots + \epsilon(x_{n-1}) \quad \text{and}
\]

\[
\frac{1}{nx_n^a} = \frac{1}{nx_1^a} = \frac{n - 1}{nk^a} + \epsilon(x_1) + \cdots + \epsilon(x_{n-1}) \rightarrow \frac{1}{k^a}
\]

as \( n \to \infty \). Therefore \( n^{1/a} x_n \to k \).

(b) Let \( a = 2 \). Compute

\[
\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^4} = \frac{1}{3}
\]

by L’Hospital’s rule (or Maclaurin series). Hence \( k = \sqrt{3} \).

(c) Let \( a = 1 \). Compute

\[
\lim_{x \to 0} \frac{x - (1 - e^{-x})}{x(1 - e^{-x})} = \frac{1}{2}
\]

by writing \( e^{-x} = 1 - x + x^2/2 - x^2 \omega(x) \) where \( \omega(x) \to 0 \) as \( x \to 0 \). Hence \( k = 2 \).

Editorial comment. Note that the differentiability of \( f \) is not needed.

Also solved by S. Amghibech (Canada), M. R. Avidon, O. Bagdasar (Romania), P. Bracken, D. R. Bridges, R. Chapman (U. K.), P. P. Dályay (Hungary), J.-P. Grivaux (France), J. H. Lindsey II, O. López & N. Caro (Brazil), M. McMullen, M. D. Meyerson, J. Rooin & A. Morassaei (Iran), K. Schilling, H.-J. Seiffert (Germany), N. C. Singer, A. Stadler (Switzerland), A. Stenger, R. Stong, M. Téivá (Romania), D. Văcăru (Romania), J. Vinuesa (Spain), Z. Vörös (Hungary), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, Northwestern University Math Problem Solving Group, and the proposers.