EXERCISES

Let G be a group and let Ω be an infinite set.

1. Prove that A_n does not have a proper subgroup of index $< n$ for all $n \geq 5$.

2. Find all normal subgroups of S_n for all $n \geq 5$.

3. Prove that A_n is the only proper subgroup of index $< n$ in S_n for all $n \geq 5$.

4. Prove that A_n is generated by the set of all 3-cycles for each $n \geq 3$.

5. Prove that if there exists a chain of subgroups $G_1 \leq G_2 \leq \ldots \leq G$ such that $G = \cup_{i=1}^{\infty} G_i$ and each G_i is simple then G is simple.

6. Let D be the subgroup of S_Ω consisting of permutations which move only a finite number of elements of Ω (described in Exercise 17 in Section 3) and let A be the set of all elements $\sigma \in D$ such that σ acts as an even permutation on the (finite) set of points it moves. Prove that A is an infinite simple group. [Show that every pair of elements of D lie in a finite simple subgroup of D.]

7. Under the notation of the preceding, exercise prove that if $H \trianglelefteq S_\Omega$ and $H \neq 1$ then $A \leq H$, i.e., A is the unique (nontrivial) minimal normal subgroup of S_Ω.

8. Under the notation of the preceding two exercises prove that $|D| = |A| = |\Omega|$. Deduce that

$$\text{if } S_\Omega \cong S_\Delta \text{ then } |\Omega| = |\Delta|.$$

[Use the fact that D is generated by transpositions. You may assume that countable unions and finite direct products of sets of cardinality $|\Omega|$ also have cardinality $|\Omega|$.]