above) the kernel of this representation is a proper normal subgroup and G is simple, the kernel is 1 and G is isomorphic to a subgroup of S_5. Identify G with this isomorphic copy so that we may assume $G \leq S_5$. If G is not contained in A_5, then $S_5 = GA_5$ and, by the Second Isomorphism Theorem, $A_5 \cap G$ is of index 2 in G. Since G has no (normal) subgroup of index 2, this is a contradiction. This argument proves $G \leq A_5$. Since $|G| = |A_5|$, the isomorphic copy of G in S_5 coincides with A_5, as desired.

Finally, assume $n_2 = 15$. If for every pair of distinct Sylow 2-subgroups P and Q of G, $P \cap Q = 1$, then the number of nonidentity elements in Sylow 2-subgroups of G would be $(4-1) \cdot 15 = 45$. But $n_5 = 6$ so the number of elements of order 5 in G is $(5-1) \cdot 6 = 24$, accounting for 69 elements. This contradiction proves that there exist distinct Sylow 2-subgroups P and Q with $|P \cap Q| = 2$. Let $M = N_G(P \cap Q)$. Since P and Q are abelian (being groups of order 4), P and Q are subgroups of M and since G is simple, $M \neq G$. Thus 4 divides $|M|$ and $|M| > 4$ (otherwise, $P = M = Q$). The only possibility is $|M| = 12$, i.e., M has index 5 in G (recall M cannot have index 3 or 1). But now the argument of the preceding paragraph applied to M in place of N gives $G \cong A_5$. This leads to a contradiction in this case because $n_2(A_5) = 5$ (cf. the exercises). The proof is complete.

EXERCISES

Let G be a finite group and let p be a prime.

1. Prove that if $P \in Syl_p(G)$ and H is a subgroup of G containing P then $P \in Syl_p(H)$. Give an example to show that, in general, a Sylow p-subgroup of a subgroup of G need not be a Sylow p-subgroup of G.
2. Prove that if H is a subgroup of G and $Q \in Syl_p(H)$ then $gQg^{-1} \in Syl_p(gHg^{-1})$ for all $g \in G$.
3. Use Sylow's Theorem to prove Cauchy's Theorem. (Note that we only used Cauchy's Theorem for abelian groups — Proposition 3.21 — in the proof of Sylow's Theorem so this line of reasoning is not circular.)
4. Exhibit all Sylow 2-subgroups and Sylow 3-subgroups of D_{12} and $S_3 \times S_3$.
5. Show that a Sylow p-subgroup of D_{2n} is cyclic and normal for every odd prime p.
6. Exhibit all Sylow 3-subgroups of A_4 and all Sylow 3-subgroups of S_4.
7. Exhibit all Sylow 2-subgroups of S_4 and find elements of S_4 which conjugate one of these into each of the others.
8. Exhibit two distinct Sylow 2-subgroups of S_5 and an element of S_5 that conjugates one into the other.
9. Exhibit all Sylow 3-subgroups of $SL_2(\mathbb{F}_3)$ (cf. Exercise 9, Section 2.1).
10. Prove that the subgroup of $SL_2(\mathbb{F}_3)$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the unique Sylow 2-subgroup of $SL_2(\mathbb{F}_3)$ (cf. Exercise 10, Section 2.4).
11. Show that the center of $SL_2(\mathbb{F}_3)$ is the group of order 2 consisting of $\pm I$, where I is the identity matrix. Prove that $SL_2(\mathbb{F}_3)/Z(SL_2(\mathbb{F}_3)) \cong A_4$. [Use facts about groups of order 12.]
12. Let $2n = 2^a k$ where k is odd. Prove that the number of Sylow 2-subgroups of D_{2n} is k. [Prove that if $P \in Syl_2(D_{2n})$ then $N_{D_{2n}}(P) = P$.]

13. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

14. Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

15. Prove that a group of order 351 has a normal Sylow p-subgroup for some prime p dividing its order.

16. Let $|G| = pqr$, where p, q and r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q or r.

17. Prove that if $|G| = 105$ then G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.

18. Prove that a group of order 200 has a normal Sylow 5-subgroup.

19. Prove that if $|G| = 6545$ then G is not simple.

20. Prove that if $|G| = 1365$ then G is not simple.

21. Prove that if $|G| = 2907$ then G is not simple.

22. Prove that if $|G| = 132$ then G is not simple.

23. Prove that if $|G| = 462$ then G is not simple.

24. Prove that if G is a group of order 231 then $Z(G)$ contains a Sylow 11-subgroup of G and a Sylow 7-subgroup is normal in G.

25. Prove that if G is a group of order 385 then $Z(G)$ contains a Sylow 7-subgroup of G and a Sylow 11-subgroup is normal in G.

26. Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal then G is abelian.

27. Let G be a group of order 315 which has a normal Sylow 3-subgroup. Prove that $Z(G)$ contains a Sylow 3-subgroup of G and deduce that G is abelian.

28. Let G be a group of order 1575. Prove that if a Sylow 3-subgroup of G is normal then a Sylow 5-subgroup and a Sylow 7-subgroup are normal. In this situation prove that G is abelian.

29. If G is a non-abelian simple group of order < 100, prove that $G \cong A_5$. [Eliminate all orders but 60.]

30. How many elements of order 7 must there be in a simple group of order 168?

31. For $p = 2, 3$ and 5 find $n_p(A_5)$ and $n_p(S_5)$. [Note that $A_4 \leq A_5$.]

32. Let P be a Sylow p-subgroup of H and let H be a subgroup of K. If $P \trianglelefteq H$ and $H \trianglelefteq K$, prove that P is normal in K. Deduce that if $P \in Syl_p(G)$ and $H = N_G(P)$, then $N_G(H) = H$ (in words: *normalizers of Sylow p-subgroups are self-normalizing*).

33. Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

34. Let $P \in Syl_p(G)$ and assume $N \trianglelefteq G$. Use the conjugacy part of Sylow's Theorem to prove that $P \cap N$ is a Sylow p-subgroup of N. Deduce that PN/N is a Sylow p-subgroup of G/N (note that this may also be done by the Second Isomorphism Theorem — cf. Exercise 9, Section 3.3).

35. Let $P \in Syl_p(G)$ and let $H \leq G$. Prove that $gPg^{-1} \cap H$ is a Sylow p-subgroup of H for some $g \in G$. Give an explicit example showing that $hPh^{-1} \cap H$ is not necessarily a Sylow p-subgroup of H for any $h \in H$ (in particular, we cannot always take $g = 1$ in the first part of this problem, as we could when H was normal in G).

36. Prove that if N is a normal subgroup of G then $n_p(G/N) \leq n_p(G)$.

37. Let R be a normal p-subgroup of G (not necessarily a Sylow subgroup).
   (a) Prove that R is contained in every Sylow p-subgroup of G.
   (b) If S is another normal p-subgroup of G, prove that RS is also a normal p-subgroup of G.
   (c) The subgroup $O_p(G)$ is defined to be the group generated by all normal p-subgroups of G. Prove that $O_p(G)$ is the unique largest normal p-subgroup of G and $O_p(G)$ equals the intersection of all Sylow p-subgroups of G.
   (d) Let $\overline{G} = G/O_p(G)$. Prove that $O_p(\overline{G}) = \overline{1}$ (i.e., $\overline{G}$ has no nontrivial normal p-subgroup).

38. Use the method of proof in Sylow's Theorem to show that if n_p is not congruent to $1 \pmod{p^2}$ then there are distinct Sylow p-subgroups P and Q of G such that $|P : P \cap Q| = |Q : P \cap Q| = p$.

39. Show that the subgroup of strictly upper triangular matrices in $GL_n(\mathbb{F}_p)$ (cf. Exercise 17, Section 2.1) is a Sylow p-subgroup of this finite group. [Use the order formula in Section 1.4 to find the order of a Sylow p-subgroup of $GL_n(\mathbb{F}_p)$.]

40. Prove that the number of Sylow p-subgroups of $GL_2(\mathbb{F}_p)$ is $p+1$. [Exhibit two distinct Sylow p-subgroups.]

41. Prove that $SL_2(\mathbb{F}_4) \cong A_5$ (cf. Exercise 9, Section 2.1 for the definition of $SL_2(\mathbb{F}_4)$).

42. Prove that the group of rigid motions in $\mathbb{R}^3$ of an icosahedron is isomorphic to A_5. [Recall that the order of this group is 60: Exercise 13, Section 1.2.]

43. Prove that the group of rigid motions in $\mathbb{R}^3$ of a dodecahedron is isomorphic to A_5. (As with the cube and the tetrahedron, the icosahedron and the dodecahedron are dual solids.) [Recall that the order of this group is 60: Exercise 12, Section 1.2.]

44. Let p be the smallest prime dividing the order of the finite group G. If $P \in Syl_p(G)$ and P is cyclic prove that $N_G(P) = C_G(P)$.

45. Find generators for a Sylow p-subgroup of S_{2p}, where p is an odd prime. Show that this is an abelian group of order p^2.

46. Find generators for a Sylow p-subgroup of S_{p^2}, where p is a prime. Show that this is a non-abelian group of order p^{p+1}.

47. Write and execute a computer program which
   (i) gives each odd number $n < 10,000$ that is not a power of a prime and that has some prime divisor p such that n_p is not forced to be 1 for all groups of order n by the congruence condition of Sylow's Theorem, and
   (ii) gives for each n in (i) the factorization of n into prime powers and gives the list of all permissible values of n_p for all primes p dividing n (i.e., those values not ruled out by Part 3 of Sylow's Theorem).

48. Carry out the same process as in the preceding exercise for all even numbers less than 1000. Explain the relative lengths of the lists versus the number of integers tested.

49. Prove that if $|G| = 2^n m$ where m is odd and G has a cyclic Sylow 2-subgroup then G has a normal subgroup of order m. [Use induction and Exercises 11 and 12 in Section 2.]

50. Prove that if U and W are normal subsets of a Sylow p-subgroup P of G then U is conjugate to W in G if and only if U is conjugate to W in $N_G(P)$. Deduce that two elements in the center of P are conjugate in G if and only if they are conjugate in $N_G(P)$. (A subset U of P is normal in P if $N_P(U) = P$.)

51. Let P be a Sylow p-subgroup of G and let M be any subgroup of G which contains $N_G(P)$. Prove that $|G : M| \equiv 1 (\text{mod } p)$.

The following sequence of exercises leads to the classification of all numbers n with the property that every group of order n is cyclic (for example, $n = 15$ is such an integer). These arguments are a vastly simplified prototype for the proof that every group of odd order is solvable in the sense that they use the *structure* (commutativity) of the proper subgroups and their *embedding* in the whole group (we shall see that distinct maximal subgroups intersect in the identity) to obtain a contradiction by counting arguments. In the proof that groups of odd order are solvable one uses induction to reduce to the situation in which a minimal counterexample is a simple group — but here every proper subgroup is solvable (not abelian as in our situation). The analysis of the structure and embedding of the maximal subgroups in this situation is much more complicated and the counting arguments are (roughly speaking) replaced by character theory arguments (as will be discussed in Part VI).

52. Suppose G is a finite simple group in which every proper subgroup is abelian. If M and N are distinct maximal subgroups of G prove $M \cap N = 1$. [See Exercise 23 in Section 3.]
53. Use the preceding exercise to prove that if G is any non-abelian group in which every proper subgroup is abelian then G is not simple. [Let G be a counterexample to this assertion and use Exercise 24 in Section 3 to show that G has more than one conjugacy class of maximal subgroups. Use the method of Exercise 23 in Section 3 to count the elements which lie in all conjugates of M and N, where M and N are nonconjugate maximal subgroups of G; show that this gives more than $|G|$ elements.]
54. Prove the following classification: if G is a finite group of order $p_1 p_2 \dots p_r$ where the p_i's are distinct primes such that p_i does not divide $p_j - 1$ for all i and j, then G is cyclic. [By induction, every proper subgroup of G is cyclic, so G is not simple by the preceding exercise. If N is a nontrivial proper normal subgroup, N is cyclic and G/N acts as automorphisms of N. Use Proposition 16 to show that $N \leq Z(G)$ and use induction to show $G/Z(G)$ is cyclic, hence G is abelian by Exercise 36 of Section 3.1.]
55. Prove the converse to the preceding exercise: if $n \geq 2$ is an integer such that every group of order n is cyclic, then $n = p_1 p_2 \dots p_r$ is a product of distinct primes and p_i does not divide $p_j - 1$ for all i, j. [If n is not of this form, construct noncyclic groups of order n using direct products of noncyclic groups of order p^2 and pq, where $p \mid q - 1$.]
56. If G is a finite group in which every proper subgroup is abelian, show that G is solvable.

4.6 THE SIMPLICITY OF A_n

There are a number of proofs of the simplicity of A_n, $n \geq 5$. The most elementary involves showing A_n is generated by 3-cycles. Then one shows that a normal subgroup must contain one 3-cycle hence must contain all the 3-cycles so cannot be a proper subgroup. We include a less computational approach.

Note that A_3 is an abelian simple group and that A_4 is not simple ($n_2(A_4) = 1$).

Theorem 24. A_n is simple for all $n \geq 5$.

Proof: By induction on n. The result has already been established for $n = 5$, so assume $n \geq 6$ and let $G = A_n$. Assume there exists $H \trianglelefteq G$ with $H \neq 1$ or G.