EXERCISES

Let $G$ be a group.
1. Suppose $G$ has a left action on a set $A$, denoted by $g \cdot a$ for all $g \in G$ and $a \in A$. Denote the corresponding right action on $A$ by $a \cdot g$. Prove that the (equivalence) relations $\sim$ and $\sim'$ defined by

$$a \sim b \quad \text{if and only if} \quad a = g \cdot b \quad \text{for some } g \in G$$

and

$$a \sim' b \quad \text{if and only if} \quad a = b \cdot g \quad \text{for some } g \in G$$

are the same relation (i.e., $a \sim b$ if and only if $a \sim' b$).

2. Find all conjugacy classes and their sizes in the following groups:
   (a) $D_8$  (b) $Q_8$  (c) $A_4$.

3. Find all the conjugacy classes and their sizes in the following groups:
   (a) $Z_2 \times S_3$  (b) $S_3 \times S_3$  (c) $Z_3 \times A_4$.

4. Prove that if $S \subseteq G$ and $g \in G$ then $gN_G(S)g^{-1} = N_G(gSg^{-1})$ and $gC_G(S)g^{-1} = C_G(gSg^{-1})$.

5. If the center of $G$ is of index $n$, prove that every conjugacy class has at most $n$ elements.

6. Assume $G$ is a non-abelian group of order 15. Prove that $Z(G) = 1$. Use the fact that $(g) \leq C_G(g)$ for all $g \in G$ to show that there is at most one possible class equation for $G$. [Use Exercise 36, Section 3.1.]

7. For $n = 3, 4, 6$ and 7 make lists of the partitions of $n$ and give representatives for the corresponding conjugacy classes of $S_n$.

8. Prove that $Z(S_n) = 1$ for all $n \geq 3$.

9. Show that $|C_{S_n}((1\ 2\ (3\ 4))| = 8 \cdot (n - 4)!$ for all $n \geq 4$. Determine the elements in this centralizer explicitly.

10. Let $\sigma$ be the 5-cycle $(1\ 2\ 3\ 4\ 5)$ in $S_5$. In each of (a) to (c) find an explicit element $\tau \in S_5$ which accomplishes the specified conjugation:
   (a) $\tau \sigma \tau^{-1} = \sigma^2$
   (b) $\tau \sigma \tau^{-1} = \sigma^{-1}$
   (c) $\tau \sigma \tau^{-1} = \sigma^{-2}$.

11. In each of (a) – (d) determine whether $\sigma_1$ and $\sigma_2$ are conjugate. If they are, give an explicit permutation $\tau$ such that $\tau \sigma_1 \tau^{-1} = \sigma_2$.
   (a) $\sigma_1 = (1\ 2)(3\ 4\ 5)$ and $\sigma_2 = (1\ 2\ 3)(4\ 5)$
   (b) $\sigma_1 = (1\ 5)(3\ 7\ 2)(10\ 6\ 8\ 11)$ and $\sigma_2 = (3\ 7\ 5\ 10)(4\ 9)(13\ 11\ 2)$
   (c) $\sigma_1 = (1\ 5)(3\ 7\ 2)(10\ 6\ 8\ 11)$ and $\sigma_2 = \sigma_1^3$
   (d) $\sigma_1 = (1\ 3)(2\ 4\ 6)$ and $\sigma_2 = (3\ 5)(2\ 4)(5\ 6)$.

12. Find a representative for each conjugacy class of elements of order 4 in $S_8$ and in $S_{12}$.

13. Find all finite groups which have exactly two conjugacy classes.

14. In Exercise 1 of Section 2 two labellings of the elements $\{1, a, b, c\}$ of the Klein 4-group $V$ were chosen to give two versions of the left regular representation of $V$ into $S_4$. Let $\pi_1$ be the version of regular representation obtained in part (a) of that exercise and let $\pi_2$ be the version obtained via the labelling in part (b). Let $\tau = (2\ 4)$. Show that $\tau \circ \pi_1(g) \circ \tau^{-1} = \pi_2(g)$ for each $g \in V$ (i.e., conjugation by $\tau$ sends the image of $\pi_1$ to the image of $\pi_2$ elementwise).
15. Find an element of \( S_8 \) which conjugates the subgroup of \( S_8 \) obtained in part (a) of Exercise 3, Section 2 to the subgroup of \( S_8 \) obtained in part (b) of that same exercise (both of these subgroups are isomorphic to \( D_8 \)).

16. Find an element of \( S_4 \) which conjugates the subgroup of \( S_4 \) obtained in part (a) of Exercise 5, Section 2 to the subgroup of \( S_4 \) obtained in part (b) of that same exercise (both of these subgroups are isomorphic to \( D_8 \)).

17. Let \( A \) be a nonempty set and let \( X \) be any subset of \( S_A \). Let

\[ F(X) = \{ a \in A \mid \sigma(a) = a \text{ for all } \sigma \in X \} \]

— the fixed set of \( X \).

Let \( M(X) = A - F(X) \) be the elements which are moved by some element of \( X \). Let
\( D = \{ \sigma \in S_A \mid |M(\sigma)| < \infty \} \). Prove that \( D \) is a normal subgroup of \( S_A \).

18. Let \( A \) be a set, let \( H \) be a subgroup of \( S_A \) and let \( F(H) \) be the fixed points of \( H \) on \( A \) as defined in the preceding exercise. Prove that if \( r \in N_{S_A}(H) \) then \( r \) stabilizes the set \( F(H) \) and its complement \( A - F(H) \).

19. Assume \( H \) is a normal subgroup of \( G \), \( K \) is a conjugacy class of \( G \) contained in \( H \) and \( x \in K \). Prove that \( K \) is a union of \( k \) conjugacy classes of equal size in \( H \), where \( k = |G : HCG(x)| \). Deduce that a conjugacy class in \( S_n \) which consists of even permutations is either a single conjugacy class under the action of \( A_n \) or is a union of two classes of the same size in \( A_n \). [Let \( \Lambda = CG(x) \) and \( B = H \) so \( A \cap B = C_H(x) \). Draw the lattice diagram associated to the Second Isomorphism Theorem and interpret the appropriate indices. See also Exercise 9, Section 1.]

20. Let \( \sigma \in A_n \). Show that all elements in the conjugacy class of \( \sigma \) in \( S_n \) (i.e., all elements of the same cycle type as \( \sigma \)) are conjugate in \( A_n \) if and only if \( \sigma \) commutes with an odd permutation. [Use the preceding exercise.]

21. Let \( K \) be a conjugacy class in \( S_n \) and assume that \( K \subseteq A_n \). Show \( \sigma \in S_n \) does not commute with any odd permutation if and only if the cycle type of \( \sigma \) consists of distinct odd integers. Deduce that \( K \) consists of two conjugacy classes in \( A_n \) if and only if the cycle type of an element of \( K \) consists of distinct odd integers. [Assume first that \( \sigma \in K \) does not commute with any odd permutation. Observe that \( \sigma \) commutes with each individual cycle in its cycle decomposition — use this to show that all its cycles must be of odd length. If two cycles have the same odd length, \( k \), find a product of \( k \) transpositions which interchanges them and commutes with \( \sigma \). Conversely, if the cycle type of \( \sigma \) consists of distinct integers, prove that \( \sigma \) commutes only with the group generated by the cycles in its cycle decomposition.]

22. Show that if \( n \) is odd then the set of all \( n \)-cycles consists of two conjugacy classes of equal size in \( A_n \).

23. Recall (cf. Exercise 16, Section 2.4) that a proper subgroup \( M \) of \( G \) is called maximal if whenever \( M \leq H \leq G \), either \( H = M \) or \( H = G \). Prove that if \( M \) is a maximal subgroup of \( G \) then either \( N_G(M) = M \) or \( N_G(M) = G \). Deduce that if \( M \) is a maximal subgroup of \( G \) that is not normal in \( G \) then the number of nonidentity elements of \( G \) that are contained in conjugates of \( M \) is at most \(|[M]-1||G : M|\).

24. Assume \( H \) is a proper subgroup of the finite group \( G \). Prove \( G \neq \cup_{g \in G} gHg^{-1} \), i.e., \( G \) is not the union of the conjugates of any proper subgroup. [Put \( H \) in some maximal subgroup and use the preceding exercise.]

25. Let \( G = GL_2(\mathbb{C}) \) and let \( H = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C}, \ ac \neq 0 \} \). Prove that every element of \( G \) is conjugate to some element of the subgroup \( H \) and deduce that \( G \) is the union of
conjugates of $H$. [Show that every element of $GL_2(\mathbb{C})$ has an eigenvector.]

26. Let $G$ be a transitive permutation group on the finite set $A$ with $|A| > 1$. Show that there is some $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in A$ (such an element $\sigma$ is called fixed point free).

27. Let $g_1, g_2, \ldots, g_r$ be representatives of the conjugacy classes of the finite group $G$ and assume these elements pairwise commute. Prove that $G$ is abelian.

28. Let $p$ and $q$ be primes with $p < q$. Prove that a non-abelian group $G$ of order $pq$ has a nonnormal subgroup of index $q$, so that there exists an injective homomorphism into $S_q$. Deduce that $G$ is isomorphic to a subgroup of the normalizer in $S_q$ of the cyclic group generated by the $q$-cycle $(12\ldots q)$.

29. Let $p$ be a prime and let $G$ be a group of order $p^\alpha$. Prove that $G$ has a subgroup of order $p^\beta$, for every $\beta$ with $0 \leq \beta \leq \alpha$. [Use Theorem 8 and induction on $\alpha$.]

30. If $G$ is a group of odd order, prove for any nonidentity element $x \in G$ that $x$ and $x^{-1}$ are not conjugate in $G$.

31. Using the usual generators and relations for the dihedral group $D_{2n}$ (cf. Section 1.2) show that for $n = 2k$ an even integer the conjugacy classes in $D_{2n}$ are the following: $\{1\}$, $\{r^k\}$, $\{r^{\pm 1}\}$, $\{r^{\pm 2}\}$, $\ldots$, $\{r^{\pm (k-1)}\}$, $\{sr^{2b} | b = 1, \ldots, k\}$ and $\{sr^{2b-1} | b = 1, \ldots, k\}$. Give the class equation for $D_{2n}$.

32. For $n = 2k + 1$ an odd integer show that the conjugacy classes in $D_{2n}$ are $\{1\}$, $\{r^{\pm 1}\}$, $\{r^{\pm 2}\}$, $\ldots$, $\{r^{\pm k}\}$, $\{sr^{b} | b = 1, \ldots, n\}$. Give the class equation for $D_{2n}$.

33. This exercise gives a formula for the size of each conjugacy class in $S_n$. Let $\sigma$ be a permutation in $S_n$ and let $m_1, m_2, \ldots, m_s$ be the distinct integers which appear in the cycle type of $\sigma$ (including 1-cycles). For each $i \in \{1, 2, \ldots, s\}$, assume $\sigma$ has $k_i$ cycles of length $m_i$ (so that $\sum_{i=1}^{s} k_i m_i = n$). Prove that the number of conjugates of $\sigma$ is

$$\frac{n!}{(k_1!m_1^{k_1})(k_2!m_2^{k_2})\ldots(k_s!m_s^{k_s})}.$$

[See Exercises 6 and 7 in Section 1.3 where this formula was given in some special cases.]

34. Prove that if $p$ is a prime and $P$ is a subgroup of $S_p$ of order $p$, then $|N_{S_p}(P)| = p(p-1)$. [Argue that every conjugate of $P$ contains exactly $p - 1$ $p$-cycles and use the formula for the number of $p$-cycles to compute the index of $N_{S_p}(P)$ in $S_p$.]

35. Let $p$ be a prime. Find a formula for the number of conjugacy classes of elements of order $p$ in $S_n$ (using the greatest integer function).

36. Let $\pi : G \rightarrow S_G$ be the left regular representation afforded by the action of $G$ on itself by left multiplication. For each $g \in G$ denote the permutation $\pi(g)$ by $\sigma_g$, so that $\sigma_g(x) = gx$ for all $x \in G$. Let $\lambda : G \rightarrow S_G$ be the permutation representation afforded by the corresponding right action of $G$ on itself, and for each $h \in G$ denote the permutation $\lambda(h)$ by $\tau_h$. Thus $\tau_h(x) = xh^{-1}$ for all $x \in G$ ($\lambda$ is called the right regular representation of $G$). (a) Prove that $\sigma_g$ and $\tau_h$ commute for all $g, h \in G$. (Thus the centralizer in $S_G$ of $\pi(G)$ contains the subgroup $\lambda(G)$, which is isomorphic to $G$).

(b) Prove that $\sigma_g = \tau_g$ if and only if $g$ is an element of order 1 or 2 in the center of $G$.

(c) Prove that $\sigma_g = \tau_h$ if and only if $g$ and $h$ lie in the center of $G$. Deduce that $\pi(G) \cap \lambda(G) = \pi(Z(G)) = \lambda(Z(G))$. 

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Chap. 4  Group Actions