Thus $k \mid \frac{p!}{p} = (p-1)!$. But all prime divisors of $(p-1)!$ are less than p and by the minimality of p, every prime divisor of k is greater than or equal to p. This forces $k = 1$, so $H = K \trianglelefteq G$, completing the proof.

EXERCISES

Let G be a group and let H be a subgroup of G.

1. Let $G = \{1, a, b, c\}$ be the Klein 4-group whose group table is written out in Section 2.5.

(a) Label $1, a, b, c$ with the integers 1,2,4,3, respectively, and prove that under the left regular representation of G into S_4 the nonidentity elements are mapped as follows:

$$a \mapsto (1\ 2)(3\ 4) \qquad b \mapsto (1\ 4)(2\ 3) \qquad c \mapsto (1\ 3)(2\ 4).$$

(b) Relabel $1, a, b, c$ as 1,4,2,3, respectively, and compute the image of each element of G under the left regular representation of G into S_4. Show that the image of G in S_4 under this labelling is the same *subgroup* as the image of G in part (a) (even though the nonidentity elements individually map to different permutations under the two different labellings).

2. List the elements of S_3 as 1, $(1\ 2)$, $(2\ 3)$, $(1\ 3)$, $(1\ 2\ 3)$, $(1\ 3\ 2)$ and label these with the integers 1,2,3,4,5,6 respectively. Exhibit the image of each element of S_3 under the left regular representation of S_3 into S_6.

3. Let r and s be the usual generators for the dihedral group of order 8.

(a) List the elements of D_8 as $1, r, r^2, r^3, s, sr, sr^2, sr^3$ and label these with the integers $1, 2, \dots, 8$ respectively. Exhibit the image of each element of D_8 under the left regular representation of D_8 into S_8.

(b) Relabel this same list of elements of D_8 with the integers $1, 3, 5, 7, 2, 4, 6, 8$ respectively and recompute the image of each element of D_8 under the left regular representation with respect to this new labelling. Show that the two subgroups of S_8 obtained in parts (a) and (b) are different.

4. Use the left regular representation of Q_8 to produce two elements of S_8 which generate a subgroup of S_8 isomorphic to the quaternion group Q_8.

5. Let r and s be the usual generators for the dihedral group of order 8 and let $H = \langle s \rangle$. List the left cosets of H in D_8 as $1H, rH, r^2H$ and r^3H.

(a) Label these cosets with the integers 1,2,3,4, respectively. Exhibit the image of each element of D_8 under the representation π_H of D_8 into S_4 obtained from the action of D_8 by left multiplication on the set of 4 left cosets of H in D_8. Deduce that this representation is faithful (i.e., the elements of S_4 obtained form a subgroup isomorphic to D_8).

(b) Repeat part (a) with the list of cosets relabelled by the integers 1,3,2,4, respectively. Show that the permutations obtained from this labelling form a subgroup of S_4 that is different from the subgroup obtained in part (a).

(c) Let $K = \langle sr \rangle$, list the cosets of K in D_8 as $1K, rK, r^2K$ and r^3K, and label these with the integers 1,2,3,4. Prove that, with respect to this labelling, the image of D_8 under the representation π_K obtained from left multiplication on the cosets of K is the same *subgroup* of S_4 as in part (a) (even though the subgroups H and K are different and some of the elements of D_8 map to different permutations under the two homomorphisms).

6. Let r and s be the usual generators for the dihedral group of order 8 and let $N = \langle r^2 \rangle$. List the left cosets of N in D_8 as $1N$, rN, sN and srN. Label these cosets with the integers 1,2,3,4 respectively. Exhibit the image of each element of D_8 under the representation π_N of D_8 into S_4 obtained from the action of D_8 by left multiplication on the set of 4 left cosets of N in D_8. Deduce that this representation is not faithful and prove that $\pi_N(D_8)$ is isomorphic to the Klein 4-group.
7. Let Q_8 be the quaternion group of order 8.
   (a) Prove that Q_8 is isomorphic to a subgroup of S_8.
   (b) Prove that Q_8 is not isomorphic to a subgroup of S_n for any $n \leq 7$. [If Q_8 acts on any set A of order ≤ 7 show that the stabilizer of any point $a \in A$ must contain the subgroup $\langle -1 \rangle$.]
8. Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.
9. Prove that if p is a prime and G is a group of order p^α for some $\alpha \in \mathbb{Z}^+$, then every subgroup of index p is normal in G. Deduce that every group of order p^2 has a normal subgroup of order p.
10. Prove that every non-abelian group of order 6 has a nonnormal subgroup of order 2. Use this to classify groups of order 6. [Produce an injective homomorphism into S_3.]
11. Let G be a finite group and let $\pi : G \to S_G$ be the left regular representation. Prove that if x is an element of G of order n and $|G| = mn$, then $\pi(x)$ is a product of m n-cycles. Deduce that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $\dfrac{|G|}{|x|}$ is odd.
12. Let G and π be as in the preceding exercise. Prove that if $\pi(G)$ contains an odd permutation then G has a subgroup of index 2. [Use Exercise 3 in Section 3.3.]
13. Prove that if $|G| = 2k$ where k is odd then G has a subgroup of index 2. [Use Cauchy's Theorem to produce an element of order 2 and then use the preceding two exercises.]
14. Let G be a finite group of composite order n with the property that G has a subgroup of order k for each positive integer k dividing n. Prove that G is not simple.

4.3 GROUPS ACTING ON THEMSELVES BY CONJUGATION —THE CLASS EQUATION

In this section G is any group and we first consider G *acting on itself* (i.e., $A = G$) *by conjugation*:

$$g \cdot a = gag^{-1} \qquad \text{for all } g \in G,\ a \in G$$

where gag^{-1} is computed in the group G as usual. This definition satisfies the two axioms for a group action because

$$g_1 \cdot (g_2 \cdot a) = g_1 \cdot (g_2 a g_2^{-1}) = g_1(g_2 a g_2^{-1})g_1^{-1} = (g_1g_2)a(g_1g_2)^{-1} = (g_1g_2) \cdot a$$

and

$$1 \cdot a = 1a1^{-1} = a$$

for all $g_1, g_2 \in G$ and all $a \in G$.