GROUP ACTIONS AND PERMUTATION REPRESENTATIONS

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Let G be a group and let A be a nonempty set.

1. Let G act on the set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$(G_a is the stabilizer of a). Deduce that if G acts transitively on A then the kernel of the action is $\cap_{g\in G} gG_ag^{-1}$.
Solution. If $x \in G_b$ then we write $x = g \cdot (g^{-1}xg) \cdot g^{-1}$ and note that $(g^{-1}xg)a = g^{-1}x(ga) = g^{-1}xb = g^{-1}(xb) = g^{-1}b = a$ we'll have $x \in gG_ag^{-1}$, so $G_b \subseteq gG_ag^{-1}(*)$. If $x \in gG_ag^{-1}$ then $x = gyg^{-1}$ for some $y \in G_a$, from this we have $xb = gyg^{-1}b = gy(g^{-1}b) = gya = g(ya) = ga = b$ therefore $x \in G_b$, so $gG_ag^{-1} \subseteq G_b(**)$. From $(*)$ and $(**)$ we have $G_b = gG_ag^{-1}$. Second part of the problem is trivial because the kernel of the act is equal to $\cap_{a\in A} G_a$.

2. Let G be a permutation group on the set A(i.e., $G \leq S_A$), let $\delta \in G$ and let $a \in A$. Prove that $\delta G_a\delta^{-1} = G_{\delta(a)}$. Deduce that if G acts transitively on A then $\cap_{\delta\in G} \delta G_a\delta^{-1} = 1$.
Solution. This is a corollary of the problem 1 and definition of kernel.

3. Assume that G is an abelian, transitive supgroup of S_A. Show that $\delta(a) \neq a \forall \delta \in G - \{1\} \forall a \in A$. Deduce that $|G| = |A|$[Use the preceding exercise.]
Solution. Use the problem 2. By G is an abelian group we have $\delta G_a\delta^{-1} = G_a$, then $G_a = 1 \forall a \in A$, and we have first part. For the second part we note that by transitive , $|A|$ is equal to the number of elements of equivalence class and equal to $[G : G_a] = |G|$.(Proposition 2)

4. Let S_3 act on the set Ω of ordered pairs: $\{(i, j)|1 \leq i, j \leq 3\}$ by $\delta((i, j)) = (\delta(i), \delta(j))$. Find the orbits of S_3 on Ω. For each $\delta \in S_3$ find the cycle decomposition of δ under this action (i.e., find its cycle decomposition when δ is considered as an element of S_9- first fix a labelling of these nine ordered pairs). For each orbit $\mathcal{O}$ of S_3 acting on these nine points pick some $a \in \mathcal{O}$ and find the stabilizer of a in S_3.
Solution. It is easy to see that the orbits of S_3 on Ω are $\{(1, 1), (2, 2), (3, 3)\}$ and $\{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$. The third part of the problem is easy, for the second part, we labelling as following $(1, 1) = 1, (1, 2) = 2, (1, 3) = 3, (2, 3) = 4, (2, 2) = 5, (2, 1) = 6, (3, 1) = 7, (3, 2) = 8$ and $(3, 3) = 9$, it is easy, too.

5. For each parts (a) and (b) repeat the preceding exercise but with S_3 action on the specified set:
(a)The set of 27 triples $\{(i, j, k)|1 \leq i, j, k \leq 3\}$
(b)The set $\mathcal{P}(\{1, 2, 3\}) - \{\emptyset\}$ of all 7 nonempty subsets of $\{1, 2, 3\}$.

Date: 11.11.2008.

Solution. It is easy and similary with the problem 4.

6. Let R be the set of all polynomials with integer coefficients in the independent variables x_1, x_2, x_3, x_4 and S_4 act on R by permuting the indices of the four variables:

$$\sigma \cdot p(x_1, x_2, x_3, x_4) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$

for all $\sigma \in S_4$ and $p \in R$.
a)Find the polynomials in the orbit of S_4 on R containing $x_1 + x_2$;
b)Find the polynomials in the orbit of S_4 on R containing $x_1x_2 + x_3x_4$;
c)Find the polynomials in the orbit of S_4 on R containing $(x_1 + x_2)(x_3 + x_4)$.
Solution. Solved on the class!

7. Let G be a transitive permutation group on the finite set A. A block is a nonempty subset B of A such that for all $\sigma \in G$ either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$.
a)Prove that if B is a block containing the element a of A then $G_B := \{\sigma \in G | \sigma(B) = B\}$ is a subgroup of G containing G_a;
b)Show that if B is a block and $\sigma_1(B), \cdots, \sigma_n(B)$ are all dinstinct images of B under the elements of G then these form a partion of A;
c)A transitive group G on a set A is said to be primitive if the only blocks in A are the trivial ones: the sets of size 1 and A itself. Show that S_4 is primitive on $A = \{1, 2, 3, 4\}$. Show that D_8 is not primitive as a permutation group on the four vertices of a square;
d)Prove that the transitive group is primitive of A iff for each $a \in A$, the only subgroups of G containing G_a are G_a and G.
Solution.
a)It is easy to check that $G_B \leq G$.
If $\sigma \in G_a$ then $\sigma(a) = a$ therefore $a \in \sigma(B) \cap B$, so $\sigma(B) = B$ because B is a blocks, hence $\sigma \in G_B$.
b)We need prove following assertions
i,$\forall a \in A \exists i : a \in \sigma_i(B)$;
ii,$\sigma_i(B) \cap \sigma_j(B) = \emptyset \forall i \neq j$.
Proof of i: Choose $b \in B$ (we can because $B \neq \emptyset$) and fix it! By G is transitive, there is an element σ of G such that $a = \sigma(b)$.
Proof of ii:$\sigma_i(B) \cap \sigma_j(B) \neq \emptyset \Longrightarrow \sigma_j^{-1}\sigma_i(B) \cap B \neq \emptyset \Longrightarrow \sigma_j^{-1}\sigma_i(B) = B \Longrightarrow \sigma_i(B) = \sigma_j(B)$.
c)Routine!
d)$\Longleftarrow$: Assume that B is a block in A and $|B| > 1$, then we must prove that $B = A$. Choose $b_1, b_2 \in B$ such that $b_1 \neq b_2$ (this works because $|B| > 1$). If $G = G_B$ then $A = B$, in fact, if $A \neq B$ then we choose $a \in A, a \notin B$, because G is transitive, there is $\sigma \in G$ such that $\sigma(b_1) = a$, so $\sigma \notin G_B$. If $G = G_B \forall b \in B$ then we have contradiction, in fact, choose $\sigma \in G$ such that $\sigma(b_1) = b_2$ (by G is transitive, we can), then $\sigma \in G_B$ because B is block, therefore $\sigma \in G_b \forall b \in B$, particularly, $\sigma(b_1) = b_1$.
$\Longrightarrow$: For all $a \in A$, fix it! Assume that H is a subgroup of G such that $G_a \subset H$. We must prove that $G = H$ or $G_a = H$. On A consider following relation

$$x \sim y \Longleftrightarrow \exists h \in H : y = h(x).$$

It is easy to see that this is an equivalence relation, denote $\bar{a}$ is the class which containing a in that relation. Now, $\bar{a}$ is a block, in fact, if $\sigma \in G$ such that $\sigma(\bar{a}) \cap \bar{a} \neq \emptyset$

then $\sigma(b) = c(b, c \in \bar{a})$ therefore $\sigma(h_1(a)) = h_2(a)(h_1, h_2 \in H)$ or $h_2^{-1}\sigma h_1 \in G_a$, hence $\sigma \in H$, but by $\sigma(\bar{a}) \subset \bar{a}, |\bar{a}| < \infty$ and σ is injective we have $\sigma(\bar{a}) = \bar{a}$. Because G is primitive we have $|\bar{a}| = 1$ or $\bar{a} = A$. If $|\bar{a}| = 1$, assume that $h \in H$ then $h(a) = a$ therefore $h \in G_a$, so $H = G_a$. If $\bar{a} = A$, assume that $g \in G$ and $g(a) = b$, then $g(a) = h(a)$ for some $h \in H$ therefore $h^{-1}g \in G_a$, so $h^{-1}g \in H$, hence $g \in H$ and we are done.

8. A transitive permutation group G on a set A is called doubly transitive if for any (hence all) $a \in A$ the subgroup G_a is transitive on the set $A - \{a\}$.
a)Prove that S_n is doubly transitive on $\{1, 2, \cdots, n\}$ for all $n > 1$;
b)Prove that a doubly transitive group is primitive. Deduce that D_8 is not doubly transitive in its action on the 4 vertices of a square.
Solution.
a)Routine!
b)This is a corllary of the problem $7d$ or they are similar.

9. Assume G acts transitively on the finite set A and let H be a normal subgroup of G. Let $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_r$ be the distinct orbits of H on A.
a)Prove that G permutes the sets $\mathcal{O}_i$. Prove that G is transitive on $\{\mathcal{O}_i\}$. Deduce that all orbits of H on A have the same cardinality;
b)Prove that if $a \in \mathcal{O}_1$ then $|\mathcal{O}_1| = |H : H \cap G_a|$ and $r = |G : HG_a|$.
Solution.
a)Routine!
b)Use the Second Isomorphim Theorem and note that $G_a \cap H = H_a$.

10. Let H and K be subgroups of the group G. For each $x \in G$ define the HK double coset of x in G to be the set

$$HxK = \{hxk | h \in H, k \in K\}.$$

a)Prove that HxK is the union of the left cosets x_iK, where $\{x_iK\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K;
b)Prove that HxK is the union of right cosets of H;
c)Prove that HxK and HyK are either the same set or are disjoint for all $x, y \in G$. Show that the set of HK double cosets partitions G;
d)Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$;
e)Prove that $|HxK| = |H| \cdot |K : K \cap x^{-1}Hx|$.
Solution. Routine again!
P.S. These problems are from "Dummit and Foote, Abstract Algebra" but solutions are of mine.

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