GROUP ACTIONS AND PERMUTATION REPRESENTATIONS

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Let G be a group and let A be a nonempty set.

1. Let G act on the set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$(G_a is the stabilizer of a). Deduce that if G acts transitively on A then the kernel of the action is $\cap_{g\in G} gG_ag^{-1}$.
2. Let G be a permutation group on the set A(i.e., $G \le S_A$), let $\delta \in G$ and let $a \in A$. Prove that $\delta G_a \delta^{-1} = G_{\delta(a)}$. Deduce that if G acts transitively on A then $\cap_{\delta\in G} \delta G_a \delta^{-1} = 1$.
3. Assume that G is an abelian, transitive supgroup of S_A. Show that $\delta(a) \neq a \forall \delta \in G - \{1\} \forall a \in A$. Deduce that $|G| = |A|$[Use the preceding exercise.]
4. Let S_3 act on the set Ω of ordered pairs: $\{(i, j)|1 \le i, j \le 3\}$ by $\delta((i, j)) = (\delta(i), \delta(j))$. Find the orbits of S_3 on Ω. For each $\delta \in S_3$ find the cycle decomposition of δ under this action (i.e., find its cycle decomposition when δ is considered as an element of S_9- first fix a labelling of these nine ordered pairs). For each orbit $\mathcal{O}$ of S_3 acting on these nine points pick some $a \in \mathcal{O}$ and find the stabilizer of a in S_3.
5. For each parts (a) and (b) repeat the preceding exercise but with S_3 action on the specified set:
(a)The set of 27 triples $\{(i, j, k)|1 \le i, j, k \le 3\}$
(b)The set $\mathcal{P}(\{1, 2, 3\}) - \{\emptyset\}$ of all 7 nonempty subsets of $\{1, 2, 3\}$.
6. Let R be the set of all polynomials with integer coefficients in the independent variables x_1, x_2, x_3, x_4 and S_4 act on R by permuting the indices of the four variables:

$$\sigma \cdot p(x_1, x_2, x_3, x_4) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$

for all $\sigma \in S_4$ and $p \in R$.
a)Find the polynomials in the orbit of S_4 on R containing $x_1 + x_2$;
b)Find the polynomials in the orbit of S_4 on R containing $x_1x_2 + x_3x_4$;
c)Find the polynomials in the orbit of S_4 on R containing $(x_1 + x_2)(x_3 + x_4)$.
7. Let G be a transitive permutation group on the finite set A. A block is a nonempty subset B of A such that for all $\sigma \in G$ either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$.
a)Prove that if B is a block containing the element a of A then $G_B := \{\sigma \in G|\sigma(B) = B\}$ is a subgroup of G containing G_a;
b)Show that if B is a block and $\sigma_1(B), \cdots, \sigma_n(B)$ are all dinstinct images of B under the elements of G then these form a partion of A;
c)A transitive group G on a set A is said to be primitive if the only blocks in A are the trivial ones: the sets of size 1 and A itself. Show that S_4 is primitive on $A = \{1, 2, 3, 4\}$. Show that D_8 is not primitive as a permutation group on the four vertices of a square;
d)Prove that the transitive group is primitive of A iff for each $a \in A$, the only subgroups of G containing G_a are G_a and G.

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8. A transitive permutation group G on a set A is called doubly transitive if for any (hence all) $a \in A$ the subgroup G_a is transitive on the set $A - \{a\}$.
a)Prove that S_n is doubly transitive on $\{1, 2, \cdots, n\}$ for all $n > 1$;
b)Prove that a doubly transitive group is primitive. Deduce that D_8 is not doubly transitive in its action on the 4 vertices of a square.
9. Assume G acts transitively on the finite set A and let H be a normal subgroup of G. Let $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_r$ be the distinct orbits of H on A.
a)Prove that G permutes the sets $\mathcal{O}_i$. Prove that G is transitive on $\{\mathcal{O}_i\}$. Deduce that all orbits of H on A have the same cardinality;
b)Prove that if $a \in \mathcal{O}_1$ then $|\mathcal{O}_1| = |H : H \cap G_a|$ and $r = |G : HG_a|$.
10. Let H and K be subgroups of the group G. For each $x \in G$ define the HK double coset of x in G to be the set

$$HxK = \{hxk | h \in H, k \in K\}.$$

a)Prove that HxK is the union of the left cosets x_iK, where $\{x_iK\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K;
b)Prove that HxK is the union of right cosets of H;
c)Prove that HxK and HyK are either the same set or are disjoint for all $x, y \in G$. Show that the set of HK double cosets partitions G;
d)Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$;
e)Prove that $|HxK| = |H| \cdot |K : K \cap x^{-1}Hx|$.
P.S. These problems are from ”Dummit and Foote, Abstract Algebra”

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